Imprecise Probabilities with a Generalized Interval Form

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Abstract. Different representations of imprecise probabilities have been proposed, such as behavioral theory, evidence theory, possibility theory, probability bound analysis, F-probabilities, fuzzy probabilities, and clouds. These methods use interval-valued parameters to discribe probability distributions such that uncertainty is distinguished from variability. In this paper, we proposed a new form of imprecise probabilities based on generalized or modal intervals. Generalized intervals are algebraically closed under Kaucher arithmetic, which provides a concise representation and calculus structure as an extension of precise probabilities.

With the separation between proper and improper interval probabilities, *focal* and *non-focal* events are differentiated based on the modalities and logical semantics of generalized interval probabilities. Focal events have the semantics of critical, uncontrollable, specified, etc. in probabilistic analysis, whereas the corresponding non-focal events are complementary, controllable, and derived.

A generalized imprecise conditional probability is defined based on unconditional interval probabilities such that the algebraic relation between conditional and marginal interval probabilities is maintained. A Bayes' rule with generalized intervals (GIBR) is also proposed. The GIBR allows us to interpret the logic relationship between interval prior and posterior probabilities.

Keywords: imprecise probablity, conditioning, updating, interval arithmetic, generalized interval

1. Introduction

Imprecise probability differentiates uncertainty from variability both qualitatively and quantitatively, which is to complement the traditional sensitivity analaysis in probablistic reasoning. There have been several interval-based representations proposed in the past four decades and applied in various engineering domains, such as sensor data fusion (Guede and Girardi, 1997; Elouedi et al., 2004), reliability assessment (Kozine and Filimonov, 2000; Berleant and Zhang, 2004; Coolen, 2004), reliability-based design optimization (Mourelatos and Zhou, 2006; Du et al., 2006), design decision making under uncertainty (Nikolaidis et al., 2004; Aughenbaugh and Paredis, 2006). The core issue is to characterize incomplete knowledge with lower and upper probability pairs so that we can improve the robustness of decision making.

There are many representations of imprecise probabilities. For example, the Dempster-Shafer evidence theory (Dempster, 1967; Shafer, 1976) characterizes uncertainties as discrete probability masses associated with a power set of values. Belief-Plausibility pairs are used to measure likelihood. The behavioral imprecise probability theory (Walley, 1991) models behavioral uncertainties with the lower prevision (supremum acceptable buying price) and the upper prevision (infimum acceptable selling price). A random set (Molchanov, 2005) is a multi-valued mapping from the probability space to the value space. The possibility theory (Zadeh, 1978; Dubois and Prade, 1988) provides an alternative to represent uncertainties with Necessity-Possibility pairs. Probability bound analysis (Ferson et al., 2002) captures uncertain information with p-boxes which are pairs of lower and upper probability distributions. F-probability (Weichselberger, 2000) incorporates intervals into probability values which maintains Kolmogorov properties. Fuzzy probability (Möller and Beer, 2004) considers probability distributions with fuzzy parameters. A cloud (Neumaier, 2004) is a fuzzy interval with an interval-valued membership, which is a combination of fuzzy sets, intervals, and probability distributions.

These different representations model the indeterminacy due to incomplete information very well with different forms. There are still challenges in practical issues such as assessment and computation to derive inferences and conclusions (Walley, 1996). A simple algebraic structure is important for applications in engineering and science. In this paper, we propose a new form of imprecise probabilities based on generalized intervals. Unlike traditional set-based intervals, such as the interval $[0.1, 0.2]$ which represents a set of real values between 0.1 and 0.2, generalized or modal intervals also allow the existence of the interval $[0.2, 0.1]$. With this extension, logic quantifiers ($\forall$ and $\exists$) can be integrated to provide the interpretation of intervals. Another advantage of generalized interval is that it is closed under arithmetic operations $(+, -, \times, \div)$. This property simplifies the set structures.

We are interested to explore the potential of generalized interval to provide a connection between imprecise and precise probability, as well as among different representations of imprecise probability. In this paper, we study the algebraic properties of imprecise probablities with a generalized interval form and associated interpretation issues. In the remainder of the paper, Section 2 gives a brief overview of generalized intervals. Section 3 presents the interval probability with the generalized interval form. Section 4 describes the Bayes' rule based on generalized intervals.

2. Generalized Interval

Modal interval analysis (MIA) (Gardenes et al., 2001; Markov, 2001; Shary, 2002; Popova, 2001; Armengol et al., 2001) is an algebraic and semantic extension of interval analysis (IA) (Moore, 1966). Unlike the classical interval analysis which identifies an interval by a set of real numbers, MIA identifies the intervals by the set of predicates which is fulfilled by the real numbers. A generalized interval is not restricted to ordered bounds. A modal interval or generalized interval $\mathbf{x} := [\underline{x}, \overline{x}] \in \mathbb{KR}$ is called proper when $\underline{x} \leq \overline{x}$ and improper when $\underline{x} \geq \overline{x}$. The set of proper intervals is denoted by $\mathbb{IR} = \{[\underline{x}, \overline{x}] \mid \underline{x} \leq \overline{x}\}$, and the set of improper interval is $\overline{\mathbb{IR}} = \{[\underline{x}, \overline{x}] \mid \underline{x} \geq \overline{x}\}$. Operations are defined in Kaucher arithmetic (Kaucher, 1980).

Given a generalized interval $\mathbf{x} = [\underline{x}, \overline{x}] \in \mathbb{KR}$, two operators *pro* and *imp* return proper and improper values respectively, defined as

$$\mathrm{pro}\mathbf{x} := [\min(\underline{x}, \overline{x}), \max(\underline{x}, \overline{x})] \tag{1}$$

$$\mathrm{imp}\mathbf{x} := [\max(\underline{x}, \overline{x}), \min(\underline{x}, \overline{x})] \tag{2}$$

The relationship between proper and improper intervals is established with the operator *dual*:

$$\mathrm{dual}\mathbf{x} := [\overline{x}, \underline{x}] \tag{3}$$

Table I. The major differences between MIA and the tranditional IA

	Classical Interval Analysis	**Modal Interval Analysis**
Validity	$[3,2]$ is an invalid or empty interval	Both $[3,2]$ and $[3,2]$ are valid intervals
Semantics richness	$[2,3]+[2,4]=[4,7]$ is the only valid relation for $+$, and it only means"stack-up" and worst-case". $-,\times,\div$ are similar.	$[2,3]+[2,4]=[4,7]$, $[2,3]+[4,2]=[6,5]$, $[3,2]+[2,4]=[5,6]$, $[3,2]+[4,2]=[7,4]$ are all valid, and each has a different meaning. $-,\times,\div$ are similar.
Completeness of arithmetic	$\mathbf{a}+\mathbf{x}=\mathbf{b}$, but $\mathbf{x}\neq\mathbf{b}-\mathbf{a}$. $[2,3]+[2,4]=[4,7]$, but $[2,4]\neq[4,7]-[2,3]$ $\mathbf{a}\times\mathbf{x}=\mathbf{b}$, but $\mathbf{x}\neq\mathbf{b}\div\mathbf{a}$. $[2,3]\times[3,4]=[6,12]$, but $[3,4]\neq[6,12]\div[2,3]$ $\mathbf{x}-\mathbf{x}\neq 0$ $[2,3]-[2,3]=[-1,1]\neq 0$	$\mathbf{a}+\mathbf{x}=\mathbf{b}$, and $\mathbf{x}=\mathbf{b}-\mathrm{dual}\mathbf{a}$. $[2,3]+[2,4]=[4,7]$, and $[2,4]=[4,7]-[3,2]$ $\mathbf{a}\times\mathbf{x}=\mathbf{b}$, and $\mathbf{x}=\mathbf{b}\div\mathrm{dual}\mathbf{a}$. $[2,3]\times[3,4]=[6,12]$, and $[3,4]=[6,12]\div[3,2]$ $\mathbf{x}-\mathrm{dual}\mathbf{x}=0$ $[2,3]-[3,2]=0$

For example, $\mathbf{a}=[-1,1]$ and $\mathbf{b}=[1,-1]$ are both valid intervals. While $\mathbf{a}$ is a proper interval, $\mathbf{b}$ is an improper one. The relation between $\mathbf{a}$ and $\mathbf{b}$ can be established by $\mathbf{a}=\mathrm{dual}\mathbf{b}$. The *inclusion* relation between generalized intervals $\mathbf{x}=[\underline{x},\overline{x}]$ and $\mathbf{y}=[\underline{y},\overline{y}]$ is defined as

$$\begin{array}{l} [\underline{x},\overline{x}]\subseteq[\underline{y},\overline{y}] \iff \underline{x}\geq\underline{y}\wedge\overline{x}\leq\overline{y} \\ [\underline{x},\overline{x}]\supseteq[\underline{y},\overline{y}] \iff \underline{x}\leq\underline{y}\wedge\overline{x}\geq\overline{y} \end{array} \tag{4}$$

The *less-than-or-equal-to* and *greater-than-or-equal-to* relations are defined as

$$\begin{array}{l} [\underline{x},\overline{x}]\leq[\underline{y},\overline{y}] \iff \underline{x}\leq\underline{y}\wedge\overline{x}\leq\overline{y} \\ [\underline{x},\overline{x}]\geq[\underline{y},\overline{y}] \iff \underline{x}\geq\underline{y}\wedge\overline{x}\geq\overline{y} \end{array} \tag{5}$$

Table I lists the major differences between MIA and IA. MIA offers better algebraic properties and more semantic capabilities.

For a solution set $\mathcal{S}\subset\mathbb{R}^n$ of the interval system $\mathbf{f}(\mathbf{x})=0$ where $\mathbf{x}\in\mathbb{IR}^n$, an inner estimation $\mathbf{x}^{in}$ of the solution set $\mathcal{S}$ is an interval vector that is guaranteed to be included in the solution set, and an outer estimation $\mathbf{x}^{out}$ of $\mathcal{S}$ is an interval vector that is guaranteed to include the solution set. Not only for outer range estimations, generalized intervals are also convenient for inner range estimations (Kupriyanova, 1995; Kreinovich et al., 1996; Goldsztejn, 2005).

Another uniqueness of generalized intervals is the modal semantic extension. Unlike IA which identifies an interval by a set of real numbers only, MIA identifies an interval by a set of predicates which is fulfilled by real numbers. Given a set of closed intervals of real numbers in $\mathbb{R}$, and the set of logical existential ($\exists$) and universal ($\forall$) quantifiers, each generalized interval has an associated quantifier. The semantics of $\mathbf{x}\in\mathbb{KR}$ is denoted by $(\mathrm{Q}_{\mathbf{x}}x\in\mathrm{pro}\mathbf{x})$ where $\mathrm{Q}_{\mathbf{x}}\in\{\exists,\forall\}$. An interval $\mathbf{x}\in\mathbb{KR}$ is called *existential* if $\mathrm{Q}_{\mathbf{x}}=\exists$. Otherwise, it is called universal if $\mathrm{Q}_{\mathbf{x}}=\forall$. If a real relation

$z = f(x_1, \ldots, x_n)$ is extended to the interval relation $\mathbf{z} = \mathbf{f}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$, the interval relation $\mathbf{z}$ is interpretable if there is a semantic relation

$$(\mathrm{Q}_{\mathbf{x}_1} x_1 \in \mathrm{pro}\mathbf{x}_1) \cdots (\mathrm{Q}_{\mathbf{x}_n} x_n \in \mathrm{pro}\mathbf{x}_n)\, (\mathrm{Q}_{\mathbf{z}} z \in \mathrm{pro}\mathbf{z})\, (z = f(x_1, \ldots, x_n)) \tag{6}$$

In this paper, we propose an interval probability representation that incorporates the generalized interval in imprecise probability. The aim is to take the advantage of its algebraic closure so that the structure of interval probability can be simplified. At the same time, the interpretation of probablistic properties can be integrated with the logic relations in the structure.

3. Imprecise Probability based on Generalized Intervals

Given a sample space Ω and a σ-algebra $\mathcal{A}$ of random events over Ω, we define the generalized interval probability $\mathbf{p} : \mathcal{A} \mapsto [0,1] \times [0,1]$ which obeys the axioms of Kolmogorov: (1) $\mathbf{p}(\Omega) = [1,1] = 1$; (2) $0 \le \mathbf{p}(E) \le 1$ ($\forall E \in \mathcal{A}$); and (3) for any countable mutually disjoint events $E_i \cap E_j = \emptyset$ ($i \neq j$), $\mathbf{p}(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} \mathbf{p}(E_i)$. This implies $\mathbf{p}(\emptyset) = 0$. We also define

$$\mathbf{p}(E_1 \cup E_2) := \mathbf{p}(E_1) + \mathbf{p}(E_2) - \mathrm{dual}\mathbf{p}(E_1 \cap E_2) \tag{7}$$

When the probabilities of E_1 and E_2 are measurable and become precise, Eq.(7) has the same form as the traditional precise probabilities. The lower and upper probabilities in the generalized interval form do not have the traditional meanings of lower and upper envelopes. Rather, they provide the algebraic closure. From Eq.(7), we have

$$\mathbf{p}(E_1 \cup E_2) + \mathbf{p}(E_1 \cap E_2) = \mathbf{p}(E_1) + \mathbf{p}(E_2) \tag{8}$$

which also indicates the generalized interval probabilities are 2-monotone (and 2-alternating) in the sense of Choquet's capacities. But the relation of Eq.(8) is stronger than 2-monotonicity.

Let $(\Omega, \mathcal{A})$ be the probability space and $\mathcal{P}$ a non-empty set of probability distribution on that space. The lower and upper probability envelopes are usually defined as

$$P_*(E) = \inf_{P \in \mathcal{P}} P(E)$$
$$P^*(E) = \sup_{P \in \mathcal{P}} P(E)$$

Not every probability envelope is 2-monotone. However, 2-monotone closed-form representations are more applicable because it may be difficult to track probability envelopes during manipulations. Therefore it is of our interest that a simple algebraic structure can provide such practical advantages for broader applications.

Furthermore, we have

$$\mathbf{p}(E_1 \cup E_2) \le \mathbf{p}(E_1) + \mathbf{p}(E_2) \ (\forall E_1, E_2 \in \mathcal{A}) \tag{9}$$

in the new interval representation, since $\mathbf{p}(E_1 \cap E_2) \ge 0$. Note that Eq.(9) is different from the relation defined in the Dempster-Shafer structure or F-probability. Here it has the same form as the precise probability except for the newly defined inequality ($\le, \ge$) relations for generalized intervals. Both lower and upper probabilities are subadditive. Similar to the precise probability, the equality of Eq.(9) occurs when $\mathbf{p}(E_1 \cap E_2) = 0$.

We also define the probability of the complement of event E as

$$\mathbf{p}(E^c) := 1 - \mathrm{dual}\mathbf{p}(E) \tag{10}$$

which is equivalent to

$$\underline{p}(E^c) := 1 - \overline{p}(E) \tag{11}$$

$$\overline{p}(E^c) := 1 - \underline{p}(E) \tag{12}$$

The definitions in Eq.(11) and Eq.(12) are equivalent to the other forms of interval probabilities. The calculation based on generalized intervals as in Eq.(10) can be more concise.

$$\mathbf{p}(E) + \mathbf{p}(E^c) = 1 \ (\forall E \in \mathcal{A}) \tag{13}$$

In general, for a mutually disjoint event partition $\bigcup_{i=1}^{n} E_i = \Omega$, we have

$$\sum_{i=1}^{n} \mathbf{p}(E_i) = 1 \tag{14}$$

This requirement is more restrictive than the traditional coherence constraint (Walley, 1991). Suppose $\mathbf{p}(E_i) \in \mathbb{IR}$ (for $i = 1, \ldots, k$) and $\mathbf{p}(E_i) \in \overline{\mathbb{IR}}$ (for $i = k+1, \ldots, n$). If the range of an interval probability is defined as

$$\mathbf{p}'(E) := \mathrm{pro}\mathbf{p}(E) \tag{15}$$

Eq.(14) can be interpreted as

$$\forall p_1 \in \mathbf{p}'(E_1), \ldots, \forall p_k \in \mathbf{p}'(E_k), \exists p_{k+1} \in \mathbf{p}'(E_{k+1}), \ldots, \exists p_n \in \mathbf{p}'(E_n), \sum_{i=1}^{n} p_i = 1 \tag{16}$$

based on the interpretability principles of MIA (Gardenes et al., 2001). Therefore, we call Eq.(14) the *logic coherence constraint.*

The values of interval probabilities are between 0 and 1. As a result, the interval probabilities $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ have the following algebraic properties:

$$\begin{aligned}
\mathbf{p}_1 \leq \mathbf{p}_2 &\Leftrightarrow \mathbf{p}_1 + \mathbf{p}_3 \leq \mathbf{p}_2 + \mathbf{p}_3 \\
\mathbf{p}_1 \subseteq \mathbf{p}_2 &\Leftrightarrow \mathbf{p}_1 + \mathbf{p}_3 \subseteq \mathbf{p}_2 + \mathbf{p}_3 \\
\mathbf{p}_1 \leq \mathbf{p}_2 &\Leftrightarrow \mathbf{p}_1\mathbf{p}_3 \leq \mathbf{p}_2\mathbf{p}_3 \\
\mathbf{p}_1 \subseteq \mathbf{p}_2 &\Leftrightarrow \mathbf{p}_1\mathbf{p}_3 \subseteq \mathbf{p}_2\mathbf{p}_3
\end{aligned}$$

3.1. Focal and Non-Focal Events

We differentiate two types of events. An event E is a *focal* event if its associated semantics is universal ($\mathrm{Q}_{\mathbf{p}(E)} = \forall$). Otherwise it is a *non-focal* event if the semantics is existential ($\mathrm{Q}_{\mathbf{p}(E)} = \exists$). A focal event is an event of interest in the probabilistic analysis. The uncertainties associated with focal events are critical for the analysis of a system. In contrast, the uncertainties associated with non-focal events are "complementary" and "balancing". The corresponding non-focal event is not the focus of the assessment. The quantified uncertainties of non-focal events are derived from those of the corresponding focal events. For instance, in risk assessment, the high-consequence event of interest is the target and focus of study, such as the event of a hurricane landfall at U.S. coastline or the event of a structural failure at the half of a bridge's life expectancy, whereas the event of the hurricane landfall at Mexican coastline and the event of the structral failure when the bridge is twice as old as it was designed for may become non-focal.

In the interpretation in Eq.(16), the interval probability of a focal event E_i is proper ($\mathbf{p}(E_i) \in \mathbb{IR}$), and the interval probability of a non-focal event E_j is existential ($\mathbf{p}(E_j) \in \overline{\mathbb{IR}}$). Focal events have the semantics of critical, uncontrollable, specified in probabilistic analysis, whereas the corresponding non-focal events are complementary, controllable, and derived. The complement of a focal event is a non-focal event. For a set of mutually disjoint events, there is at least one non-focal event because of Eq.(14).

Two relations between events are defined. Event E_1 is said to be *less likely* (or *more likely*) to occur than event E_2, $E_1 \preceq E_2$ (or $E_1 \succeq E_2$), defined as

$$\begin{array}{l} E_1 \preceq E_2 \iff \mathbf{p}(E_1) \leq \mathbf{p}(E_2) \\ E_1 \succeq E_2 \iff \mathbf{p}(E_1) \geq \mathbf{p}(E_2) \end{array} \tag{17}$$

Event E_1 is said to be *less focused* (or *more focused*) than event E_2, denoted as $E_1 \sqsubseteq E_2$ (or $E_1 \sqsupseteq E_2$), defined as

$$\begin{array}{l} E_1 \sqsubseteq E_2 \iff \mathbf{p}(E_1) \subseteq \mathbf{p}(E_2) \\ E_1 \sqsupseteq E_2 \iff \mathbf{p}(E_1) \supseteq \mathbf{p}(E_2) \end{array} \tag{18}$$

LEMMA 3.1. *$E_1 \subseteq E_2 \Rightarrow E_1 \preceq E_2$.*

Proof. $E_1 \subseteq E_2 \Rightarrow \mathbf{p}(E_2) = \mathbf{p}(E_1 \cup (E_2 - E_1)) = \mathbf{p}(E_1) + \mathbf{p}(E_2 - E_1) - \text{dual}\mathbf{p}(E_1 \cap (E_2 - E_1)) \geq \mathbf{p}(E_1)$.

LEMMA 3.2. *If $E_1 \cap E_3 = \emptyset$ and $E_2 \cap E_3 = \emptyset$, $E_1 \preceq E_2 \Leftrightarrow E_1 \cup E_3 \preceq E_2 \cup E_3$, $E_1 \sqsubseteq E_2 \Leftrightarrow E_1 \cup E_3 \sqsubseteq E_2 \cup E_3$.*

Proof.
$E_1 \preceq E_2 \Leftrightarrow \mathbf{p}(E_1) \leq \mathbf{p}(E_2) \Leftrightarrow \mathbf{p}(E_1) + \mathbf{p}(E_3) \leq \mathbf{p}(E_2) + \mathbf{p}(E_3) \Leftrightarrow \mathbf{p}(E_1 \cup E_3) \leq \mathbf{p}(E_2 \cup E_3) \Leftrightarrow E_1 \cup E_3 \preceq E_2 \cup E_3$.
$E_1 \sqsubseteq E_2 \Leftrightarrow \mathbf{p}(E_1) \subseteq \mathbf{p}(E_2) \Leftrightarrow \mathbf{p}(E_1) + \mathbf{p}(E_3) \subseteq \mathbf{p}(E_2) + \mathbf{p}(E_3) \Leftrightarrow \mathbf{p}(E_1 \cup E_3) \subseteq \mathbf{p}(E_2 \cup E_3) \Leftrightarrow E_1 \cup E_3 \sqsubseteq E_2 \cup E_3$.

LEMMA 3.3. *If E_1 and E_3 are independent, and also E_2 and E_3 are independent, $E_1 \preceq E_2 \Leftrightarrow E_1 \cap E_3 \preceq E_2 \cap E_3$, $E_1 \sqsubseteq E_2 \Leftrightarrow E_1 \cap E_3 \sqsubseteq E_2 \cap E_3$.*

Proof.
$E_1 \preceq E_2 \Leftrightarrow \mathbf{p}(E_1) \leq \mathbf{p}(E_2) \Leftrightarrow \mathbf{p}(E_1)\mathbf{p}(E_3) \leq \mathbf{p}(E_2)\mathbf{p}(E_3) \Leftrightarrow \mathbf{p}(E_1 \cap E_3) \leq \mathbf{p}(E_2 \cap E_3) \Leftrightarrow E_1 \cap E_3 \preceq E_2 \cap E_3$.
$E_1 \sqsubseteq E_2 \Leftrightarrow \mathbf{p}(E_1) \subseteq \mathbf{p}(E_2) \Leftrightarrow \mathbf{p}(E_1)\mathbf{p}(E_3) \subseteq \mathbf{p}(E_2)\mathbf{p}(E_3) \Leftrightarrow \mathbf{p}(E_1 \cap E_3) \subseteq \mathbf{p}(E_2 \cap E_3) \Leftrightarrow E_1 \cap E_3 \sqsubseteq E_2 \cap E_3$.

LEMMA 3.4. *Suppose $E \cup E^c = \Omega$ and $\mathbf{p}(E) \in \mathbb{IR}$. (1) $\mathbf{p}(E) \leq \mathbf{p}(E^c)$ if $\overline{p}(E) \leq 0.5$; (2) $\mathbf{p}(E) \geq \mathbf{p}(E^c)$ if $\underline{p}(E) \geq 0.5$; (3) $\mathbf{p}(E) \supseteq \mathbf{p}(E^c)$ if $\underline{p}(E) \leq 0.5$ and $\overline{p}(E) \geq 0.5$.*

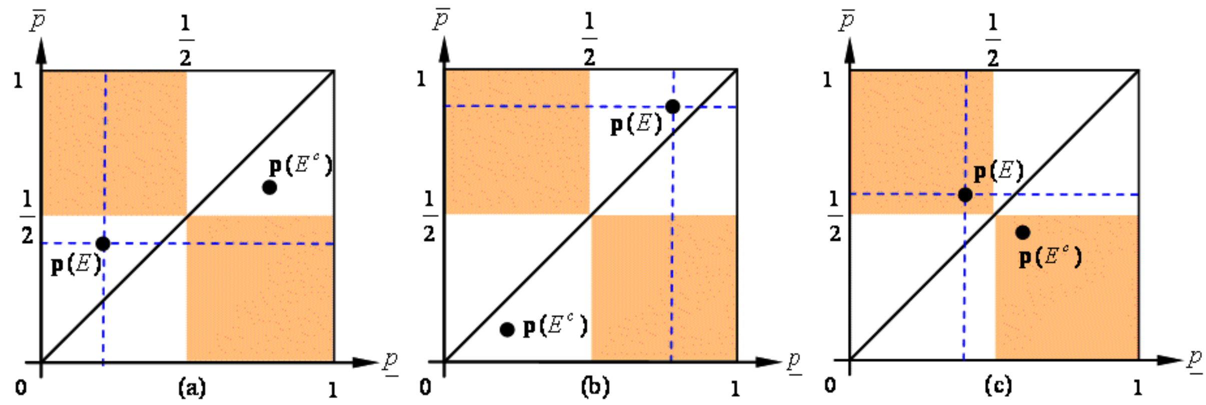

Figure 1. inf-sup diagrams for different relations between $\mathbf{p}(E)$ and $\mathbf{p}(E^c)$ when $\mathbf{p}(E) \in \mathbb{IR}$

Proof. (1) Because $\mathbf{p}(E) \in \mathbb{IR}$, $\mathbf{p}(E^c) \in \overline{\mathbb{IR}}$, and $\mathbf{p}(E) + \mathbf{p}(E^c) = 1$, it is easy to see $\underline{p}(E) \leq \underline{p}(E^c)$ and $\overline{p}(E) \leq \overline{p}(E^c)$ if $\overline{p}(E) \leq 0.5$. (2) can be verified similarly. (3) If $\underline{p}(E) \leq 0.5$ and $\overline{p}(E) \geq 0.5$, then $\underline{p}(E^c) \geq 0.5$ and $\overline{p}(E^c) \leq 0.5$. Thus $\underline{p}(E) \leq \underline{p}(E^c)$ and $\overline{p}(E) \geq \overline{p}(E^c)$.

Remark. As illustrated in Fig. 1 (a-c) respectively, a focal event E is less likely to occur than its complement if $\mathbf{p}(E) \leq 0.5$; E is more likely to occur than its complement if $\mathbf{p}(E) \geq 0.5$; otherwise, E is more focused than its complement. When E is a non-focal event, its complement E^c is a focal event. The relationships between $\mathbf{p}(E)$ and $\mathbf{p}(E^c)$ are just opposite.

For three events $E_i (i = 1, 2, 3)$,

$$\begin{aligned}\mathbf{p}(E_1 \cup E_2 \cup E_3) = \mathbf{p}(E_1) + \mathbf{p}(E_2) + \mathbf{p}(E_3) - \mathrm{dual}\mathbf{p}(E_1 \cap E_2) \\ -\mathrm{dual}\mathbf{p}(E_2 \cap E_3) - \mathrm{dual}\mathbf{p}(E_1 \cap E_3) + \mathbf{p}(E_1 \cap E_2 \cap E_3)\end{aligned}$$

In general, for $A \subseteq \Omega$,

$$\mathbf{p}(A) = \sum_{S \subseteq A} (-\mathrm{dual})^{|A|-|S|} \mathbf{p}(S) \tag{19}$$

3.2. Conditional Interval Probabilities

There have been several conditioning schemes proposed based on the Demspter-Shafer structures (Smets, 1991; Fagin and Halpern, 1991; Jaffray, 1992; Dubois and Prade, 1994; Chrisman, 1995; Kulasekere et al., 2004). Different from the coherent provision or F-probability theory, we define conditional generalized interval probabilities based on marginal probabilities. The conditional interval probability $\mathbf{p}(E|C)$ for $\forall E, C \in \mathcal{A}$ is defined as

$$\mathbf{p}(E|C) := \frac{\mathbf{p}(E \cap C)}{\mathrm{dual}\mathbf{p}(C)} = \left[\frac{\underline{p}(E \cap C)}{\underline{p}(C)}, \frac{\overline{p}(E \cap C)}{\overline{p}(C)}\right] \tag{20}$$

when $\mathbf{p}(C) > 0$.

Not only does the definition in Eq.(20) ensure the algebraic closure of the interval probability calculus, but also it is a generalization of the canonical conditional probability in F-probabilities. Different from the Dempster's rule of conditioning or geometric conditioning, this conditional structure maintains the algebraic relation between marginal and conditional probabilities. Further,

$$\mathbf{p}(C|C) = 1$$

.

The available logic interpretations of the conditional interval probabilities are as follows.

- when $\mathbf{p}(E \cap C) \in \mathbb{IR}$, $\mathbf{p}(C) \in \overline{\mathbb{IR}}$, and $\mathbf{p}(E|C) \in \mathbb{IR}$

$$\forall p_{E\cap C} \in \mathbf{p}'(E \cap C), \forall p_C \in \mathbf{p}'(C), \exists p_{E|C} \in \mathbf{p}'(E|C), p_{E|C} = \frac{p_{E\cap C}}{p_C} \tag{21}$$

or

$$\forall p_{E|C} \in \mathbf{p}'(E|C), \exists p_{E\cap C} \in \mathbf{p}'(E \cap C), \exists p_C \in \mathbf{p}'(C), p_{E|C} = \frac{p_{E\cap C}}{p_C} \tag{22}$$

- when $\mathbf{p}(E \cap C) \in \mathbb{IR}$, $\mathbf{p}(C) \in \mathbb{IR}$, and $\mathbf{p}(E|C) \in \mathbb{IR}$

$$\forall p_{E\cap C} \in \mathbf{p}'(E \cap C), \exists p_C \in \mathbf{p}'(C), \exists p_{E|C} \in \mathbf{p}'(E|C), p_{E|C} = \frac{p_{E\cap C}}{p_C} \tag{23}$$

or

$$\forall p_{E|C} \in \mathbf{p}'(E|C), \forall p_C \in \mathbf{p}'(C), \exists p_{E\cap C} \in \mathbf{p}'(E \cap C), p_{E|C} = \frac{p_{E\cap C}}{p_C} \tag{24}$$

- when $\mathbf{p}(E \cap C) \in \mathbb{IR}$, $\mathbf{p}(C) \in \mathbb{IR}$, and $\mathbf{p}(E|C) \in \overline{\mathbb{IR}}$

$$\forall p_{E\cap C} \in \mathbf{p}'(E \cap C), \forall p_{E|C} \in \mathbf{p}'(E|C), \exists p_C \in \mathbf{p}'(C), p_{E|C} = \frac{p_{E\cap C}}{p_C} \tag{25}$$

or

$$\forall p_C \in \mathbf{p}'(C), \exists p_{E\cap C} \in \mathbf{p}'(E \cap C), \exists p_{E|C} \in \mathbf{p}'(E|C), p_{E|C} = \frac{p_{E\cap C}}{p_C} \tag{26}$$

The logic interpretations of interval conditional probabilities build the connection between point measurements and probability sets. Therefore, we may use them to check if a range estimation is a tight envelope. We use the Example 3.1 in (Weichselberger, 2000) to illustrate.

EXAMPLE 3.1. *Given the following probabilities in the sample space* $\Omega = E_1 \cup E_2 \cup E_3$,

$$\begin{array}{ll} \mathbf{p}'(E_1) = [0.10, 0.25] & \mathbf{p}'(E_2 \cup E_3) = [0.75, 0.90] \\ \mathbf{p}'(E_2) = [0.20, 0.40] & \mathbf{p}'(E_1 \cup E_3) = [0.60, 0.80] \\ \mathbf{p}'(E_3) = [0.40, 0.60] & \mathbf{p}'(E_1 \cup E_2) = [0.40, 0.60] \end{array}$$

A partition of Ω *is*

$$\mathcal{C} = \{C_1, C_2\} \quad \text{where } C_1 = E_1 \cup E_2 \text{ and } C_2 = E_3$$

$$\mathbf{p}(C_1) = [0.40, 0.60] \;\; \mathbf{p}(C_2) = [0.60, 0.40]$$

Suppose $\mathbf{p}(E_1) = [0.10, 0.25]$ *and* $\mathbf{p}(C_1) = [0.60, 0.40]$*, we have*

$$\mathbf{p}(E_1|C_1) = \frac{[0.10, 0.25]}{[0.40, 0.60]} = [0.1666, 0.6250]$$

The interpretation of

$$\forall p_{E_1} \in [0.10, 0.25], \forall p_{C_1} \in [0.40, 0.60], \exists p_{E_1|C_1} \in [0.1666, 0.6250], p_{E_1|C_1} = \frac{p_{E_1}}{p_{C_1}}$$

indicates that the range estimation $\mathbf{p}(E_1|C_1) = [0.1666, 0.6250]$ *is* complete *in the sense that it considers all possible occurences of* $p(E_1)$ *and* $p(C_1)$*. However, the range estimation is not necessarily a tight envelope.*

On the other hand, if $\mathbf{p}(E_1) = [0.25, 0.10]$ *and* $\mathbf{p}(C_1) = [0.40, 0.60]$*, we have*

$$\mathbf{p}(E_1|C_1) = \frac{[0.25, 0.10]}{[0.60, 0.40]} = [0.6250, 0.1666]$$

The interpretation of

$$\forall p_{E_1|C_1} \in [0.1666, 0.6250], \exists p_{E_1} \in [0.10, 0.25], \exists p_{C_1} \in [0.40, 0.60], p_{E_1|C_1} = \frac{p_{E_1}}{p_{C_1}}$$

indicates that the range estimation $[0.1666, 0.6250]$ *is also* sound *in the sense that the range estimation is a tight envelope.*

Suppose $\mathbf{p}(E_1) = [0.25, 0.10]$, $\mathbf{p}(E_2) = [0.20, 0.40]$*, and* $\mathbf{p}(C_1) = [0.60, 0.40]$*, we have*

$$\mathbf{p}(E_1|C_1) = \frac{[0.25, 0.10]}{[0.40, 0.60]} = [0.4166, 0.25]$$

$$\mathbf{p}(E_2|C_1) = \frac{[0.20, 0.40]}{[0.40, 0.60]} = [0.3333, 1.0]$$

The interpretations are

$$\forall p_{E_1|C_1} \in [0.25, 0.4166], \forall p_{C_1} \in [0.40, 0.60], \exists p_{E_1} \in [0.10, 0.25], p_{E_1|C_1} = \frac{p_{E_1}}{p_{C_1}}$$
$$\forall p_{E_2} \in [0.20, 0.40], \forall p_{C_1} \in [0.40, 0.60], \exists p_{E_2|C_1} \in [0.3333, 1.0], p_{E_2|C_1} = \frac{p_{E_2}}{p_{C_1}}$$

respectively. Combining the two, we can have the interpretation of

$$\begin{aligned} \forall p_{E_2} \in [0.20, 0.40], \forall p_{C_1} \in [0.40, 0.60], \forall p_{E_1|C_1} \in [0.25, 0.4166], \\ \exists p_{E_1} \in [0.10, 0.25] \exists p_{E_2|C_1} \in [0.3333, 1.0], \\ p_{E_1|C_1} = \tfrac{p_{E_1}}{p_{C_1}}, p_{E_2|C_1} = \tfrac{p_{E_2}}{p_{C_1}} \end{aligned}$$

If events A and B are independent, then

$$\mathbf{p}(A|B) = \frac{\mathbf{p}(A)\mathbf{p}(B)}{\mathrm{dual}\mathbf{p}(B)} = \mathbf{p}(A) \tag{27}$$

For a mutually disjoint event partition $\bigcup_{i=1}^{n} E_i = \Omega$, we have

$$\mathbf{p}(A) = \sum_{i=1}^{n} \mathbf{p}(A|E_i)\mathbf{p}(E_i) \tag{28}$$

LEMMA 3.5. *If $B \cap C = \emptyset$, (1) $\mathbf{p}(A|C) \subseteq \mathbf{p}(A|B) \Leftrightarrow \mathbf{p}(A|B \cup C) \subseteq \mathbf{p}(A|B)$. (2) $\mathbf{p}(A|B \cup C) \supseteq \mathbf{p}(A|B) \Leftrightarrow \mathbf{p}(A|C) \supseteq \mathbf{p}(A|B)$.*

Proof. (1) $\mathbf{p}(A|C) \subseteq \mathbf{p}(A|B) \Leftrightarrow \mathbf{p}(A \cap C)/\mathrm{dual}\mathbf{p}(C) \subseteq \mathbf{p}(A|B) \Leftrightarrow \mathbf{p}(A \cap C) \subseteq \mathbf{p}(A|B)\mathbf{p}(C) \Leftrightarrow \mathbf{p}(A|B)\mathbf{p}(B) + \mathbf{p}(A \cap C) \subseteq \mathbf{p}(A|B)\mathbf{p}(B) + \mathbf{p}(A|B)\mathbf{p}(C) \Leftrightarrow \mathbf{p}(A \cap B) + \mathbf{p}(A \cap C) \subseteq \mathbf{p}(A|B)\mathbf{p}(B \cup C) \Leftrightarrow \mathbf{p}\left(A \cap (B \cup C)\right) \subseteq \mathbf{p}(A|B)\mathbf{p}(B \cup C) \Leftrightarrow \mathbf{p}\left(A \cap (B \cup C)\right)/\mathrm{dual}\mathbf{p}(B \cup C) \subseteq \mathbf{p}(A|B) \Leftrightarrow \mathbf{p}(A|B \cup C) \subseteq \mathbf{p}(A|B)$. (2) can be verified similarly.

Remark. The interpretation of the relationship (1) is that if there are two pieces of evidence (B and C), and one (C) may provide more precise estimation about a focal event (A) than the other (B) may, then the new estimation of probability about the focal event (A) based on the disjunctively combined evidence can be more precise than the one based on only one of them (B), even though the two pieces of information are contradictory to each other. The other direction of the reasoning is that if the precision of the focal event estimation with the newly introduced evidence (C) is improved, the new evidence (C) must be more informative than the old one (B) although these two are controdictory.

Remark. The interpretation of the relationship (2) is that if the estimation about a focal event (A) becomes more precise if some new evidence (B) excludes some possibilities (C) from the original evidence ($B \cup C$), then the estimation of probability about the focal event (A) based on the new evidence (B) must be more precise than the one based on the excluded one (C) along. The other direction of the reasoning is that if the precision of the focal event estimation with a contradictory evidence (C) is not improved compared to the old one with another evidence (B), then the new evidence ($B \cup C$) does not improve the estimation of the focal event (A).

4. Bayes' Rule with Generalized Intervals

The Bayes' rule with generalized intervals (GIBR) is defined as

$$\mathbf{p}(E_i|A) = \frac{\mathbf{p}(A|E_i)\mathbf{p}(E_i)}{\sum_{j=1}^{n} \mathrm{dual}\mathbf{p}(A|E_j)\mathrm{dual}\mathbf{p}(E_j)} \tag{29}$$

where $E_i (i = 1, \ldots, n)$ are mutually disjoint event partitions of Ω and $\sum_{j=1}^{n} \mathbf{p}(E_j) = 1$. The lower and upper probabilities are calculated as

$$\left[\underline{p}(E_i|A), \overline{p}(E_i|A)\right] = \left[\frac{\underline{p}(A|E_i)\underline{p}(E_i)}{\sum_{j=1}^{n} \underline{p}(A|E_j)\underline{p}(E_j)}, \frac{\overline{p}(A|E_i)\overline{p}(E_i)}{\sum_{j=1}^{n} \overline{p}(A|E_j)\overline{p}(E_j)}\right] \tag{30}$$

We can see Eq.(29) is algebraically consistent with the conditional definition in Eq.(20), with $\sum_{j=1}^{n} \mathrm{dual}\mathbf{p}(A|E_j)\mathrm{dual}\mathbf{p}(E_j) = \sum_{j=1}^{n} \mathrm{dual}\left[\mathbf{p}(A|E_j)\mathbf{p}(E_j)\right] = \mathrm{dual}\sum_{j=1}^{n} \mathbf{p}(A \cap E_j) = \mathrm{dual}\mathbf{p}(A)$.

When $n = 2$, $\mathbf{p}(E) + \mathbf{p}(E^c) = 1$. Let $\mathbf{p}(E^c) \in \overline{\mathbb{IR}}$. Eq.(29) becomes

$$\underline{p}(E|A) = \frac{\underline{p}(A|E)\underline{p}(E)}{\underline{p}(A|E)\underline{p}(E) + \underline{p}(A|E^c)\underline{p}(E^c)} = \frac{\underline{p}(A \cap E)}{\underline{p}(A \cap E) + \underline{p}(A \cap E^c)} \tag{31}$$

$$\overline{p}(E|A) = \frac{\overline{p}(A|E)\overline{p}(E)}{\overline{p}(A|E)\overline{p}(E) + \overline{p}(A|E^c)\overline{p}(E^c)} = \frac{\overline{p}(A \cap E)}{\overline{p}(A \cap E) + \overline{p}(A \cap E^c)} \tag{32}$$

When $\mathbf{p}(A \cap E) \in \mathbb{IR}$ and $\mathbf{p}(A \cap E^c) \in \overline{\mathbb{IR}}$, the relation is equivalent to the well-known 2-monotone tight envelope (Fagin and Halpern, 1991; de Campos et al., 1990; Wasserman and Kadan, 1990; Jaffray, 1992; Chrisman, 1995), given as:

$$P_*(E|A) = \frac{P_*(A \cap E)}{P_*(A \cap E) + P^*(A \cap E^c)} \tag{33}$$

$$P^*(E|A) = \frac{P^*(A \cap E)}{P^*(A \cap E) + P_*(A \cap E^c)} \tag{34}$$

where P_* and P^* are the lower and upper probability bounds defined in the traditional interval probabilities. Here $P^*(A \cap E^c) = \underline{p}(A \cap E^c)$ and $P_*(A \cap E^c) = \overline{p}(A \cap E^c)$ are the estimations of the lower and upper probability envelopes.

LEMMA 4.1. $\mathbf{p}(A|E) \subseteq \mathbf{p}(A|E^c) \Leftrightarrow \mathbf{p}(E|A) \subseteq \mathbf{p}(E)$. $\mathbf{p}(A|E) \supseteq \mathbf{p}(A|E^c) \Leftrightarrow \mathbf{p}(E|A) \supseteq \mathbf{p}(E)$.

Proof. $\mathbf{p}(A|E) \subseteq \mathbf{p}(A|E^c) \Leftrightarrow \mathbf{p}(A \cap E)/\mathrm{dual}\mathbf{p}(E) \subseteq \mathbf{p}(A \cap E^c)/\mathrm{dual}\mathbf{p}(E^c) \Leftrightarrow \mathbf{p}(A \cap E)\mathbf{p}(E^c) \subseteq \mathbf{p}(A \cap E^c)\mathbf{p}(E) \Leftrightarrow \underline{p}(A \cap E)\underline{p}(E^c) \geq \underline{p}(A \cap E^c)\underline{p}(E)$ and $\overline{p}(A \cap E)\overline{p}(E^c) \leq \overline{p}(A \cap E^c)\overline{p}(E) \Leftrightarrow \underline{p}(A \cap E)\left[1 - \underline{p}(E)\right] \geq \underline{p}(A \cap E^c)\underline{p}(E)$ and $\overline{p}(A \cap E)\left[1 - \overline{p}(E)\right] \leq \overline{p}(A \cap E^c)\overline{p}(E) \Leftrightarrow \underline{p}(A \cap E) \geq \underline{p}(A \cap E)\underline{p}(E) + \underline{p}(A \cap E^c)\underline{p}(E)$ and $\overline{p}(A \cap E) \leq \overline{p}(A \cap E)\overline{p}(E) + \overline{p}(A \cap E^c)\overline{p}(E) \Leftrightarrow \mathbf{p}(A \cap E) \subseteq \mathbf{p}(A \cap E)\mathbf{p}(E) + \mathbf{p}(A \cap E^c)\mathbf{p}(E) \Leftrightarrow \mathbf{p}(A \cap E) \subseteq \left[\mathbf{p}(A \cap E) + \mathbf{p}(A \cap E^c)\right]\mathbf{p}(E) \Leftrightarrow \mathbf{p}(A \cap E)/\mathrm{dual}\left[\mathbf{p}(A \cap E) + \mathbf{p}(A \cap E^c)\right] \subseteq \mathbf{p}(E) \Leftrightarrow \mathbf{p}(E|A) \subseteq \mathbf{p}(E)$.

The proof of $\mathbf{p}(A|E) \supseteq \mathbf{p}(A|E^c) \Leftrightarrow \mathbf{p}(E|A) \supseteq \mathbf{p}(E)$ is similar.

Remark. When the likelyhood functions $\mathbf{p}(A|E)$ and $\mathbf{p}(A|E^c)$ as well as prior and posterior probabilities are proper intervals, we can interpret the above relation as follows. If the likelyhood estimation of event A given E occurs is more accurate than that of event A given event E does not occur, then the extra information A can reduce the ambiguity of the prior estimation.

LEMMA 4.2. $\mathbf{p}(A|E) \geq \mathbf{p}(A|E^c) \Leftrightarrow \mathbf{p}(E|A) \geq \mathbf{p}(E)$. $\mathbf{p}(A|E) \leq \mathbf{p}(A|E^c) \Leftrightarrow \mathbf{p}(E|A) \leq \mathbf{p}(E)$.

Proof. The proof is similar to the previous Lemma.

Remark. If the occurance of event E increases the likelyhood estimation of event A compared to the one without the occurance of event E, then the extra information A will increase the probability of knowing that event E occurs.

LEMMA 4.3. $\mathbf{p}(A|E) = \mathbf{p}(A|E^c) \Leftrightarrow \mathbf{p}(E|A) = \mathbf{p}(E)$.

Proof. From either of the above two lemmas, $\mathbf{p}(A|E) = \mathbf{p}(A|E^c) \Leftrightarrow \mathbf{p}(A|E) \supseteq \mathbf{p}(A|E^c)$ and $\mathbf{p}(A|E) \subseteq \mathbf{p}(A|E^c) \Leftrightarrow \mathbf{p}(E|A) \supseteq \mathbf{p}(E)$ and $\mathbf{p}(E|A) \subseteq \mathbf{p}(E) \Leftrightarrow \mathbf{p}(E|A) = \mathbf{p}(E)$. Or $\mathbf{p}(A|E) =$

$\mathbf{p}(A|E^c) \Leftrightarrow \mathbf{p}(A|E) \geq \mathbf{p}(A|E^c)$ and $\mathbf{p}(A|E) \leq \mathbf{p}(A|E^c) \Leftrightarrow \mathbf{p}(E|A) \geq \mathbf{p}(E)$ and $\mathbf{p}(E|A) \leq \mathbf{p}(E) \Leftrightarrow \mathbf{p}(E|A) = \mathbf{p}(E)$.

Remark. The extra information A does not add much value to the assessment of event E if we have very similar likelyhood ratios, $\mathbf{p}(A|E)$ and $\mathbf{p}(A|E^c)$.

One of the common issues associated with the Bayes' rule based on the traditional set-based intervals is the loss of information during belief updating. The general bounds of posterior probabilities obtained depend on the sequence in which updates are performed (Pearl, 1990; Chrisman, 1995). That is, the posterior lower and upper bounds obtained by applying a series of evidences sequencially may disagree with the bounds obtained by conditioning the prior with all of the evidences in a single step. The belief updating based on Eq.(29) is sequence-independent because $\mathbf{p}(E|A)$ can be calculated incrementally, given as follows.

LEMMA 4.4. $\mathbf{p}(E|A \cap B) = \mathbf{p}(E \cap B|A)/\text{dual}\mathbf{p}(B|A)$ *for* $\forall A, B, E \in \mathcal{A}$.

Proof. $\mathbf{p}(E|A \cap B) = \mathbf{p}(E \cap A \cap B)/\text{dual}\mathbf{p}(A \cap B) = [\mathbf{p}(E \cap B|A)\mathbf{p}(A)]\,/\text{dual}\,[\mathbf{p}(B|A)\mathbf{p}(A)] = \mathbf{p}(E \cap B|A)/\text{dual}\mathbf{p}(B|A)$.

At the same time, $\mathbf{p}(E)$ can be calculated incrementally based on

$$\mathbf{p}(A \cap B) = \mathbf{p}(B|A)\mathbf{p}(A)$$

The above sequence-independent property is due to the algebraic closure of the conditional probability defined in Eq.(20).

4.1. Logic Interpretation

Some examples of logic interpretations for the relationships between prior and posterior interval probabilities in Eq.(29) are as follows.

- when $\mathbf{p}(A|E_i) \in \mathbb{IR}$, $\mathbf{p}(E_i) \in \mathbb{IR}$, $\mathbf{p}(A|E_j) \in \mathbb{IR}\,(j = 1, \ldots, n,\, j \neq i)$, $\mathbf{p}(E_{j_1}) \in \mathbb{IR}\,(j_1 = 1, \ldots, k,\, j_1 \neq i)$, $\mathbf{p}(E_{j_2}) \in \overline{\mathbb{IR}}\,(j_2 = k+1, \ldots, n,\, j_2 \neq i)$ and $\mathbf{p}(E_i|A) \in \overline{\mathbb{IR}}$

$$\begin{aligned} \forall_{j \neq i} p_{A|E_j} \in \mathbf{p}'(A|E_j), \forall_{j_1 \neq i} p_{E_{j1}} \in \mathbf{p}'(E_{j_1}), \\ \exists p_{A|E_i} \in \mathbf{p}'(A|E_i), \exists p_{E_i} \in \mathbf{p}'(E_i), \exists_{j_2 \neq i} p_{E_{j2}} \in \mathbf{p}'(E_{j2}), \exists p_{E_i|A} \in \mathbf{p}'(E_i|A), \\ p_{E_i|A} = \frac{p_{A|E_i} p_{E_i}}{\sum_{j=1}^{n} p_{A|E_j} p_{E_j}} \end{aligned} \tag{35}$$

- when $\mathbf{p}(A|E_i) \in \overline{\mathbb{IR}}$, $\mathbf{p}(E_i) \in \overline{\mathbb{IR}}$, $\mathbf{p}(A|E_j) \in \overline{\mathbb{IR}}\,(j = 1, \ldots, n,\, j \neq i)$, $\mathbf{p}(E_j) \in \overline{\mathbb{IR}}\,(j = 1, \ldots, n,\, j \neq i)$, and $\mathbf{p}(E_i|A) \in \overline{\mathbb{IR}}$

$$\begin{aligned} \forall_{j \neq i} p_{A|E_j} \in \mathbf{p}'(A|E_j), \forall_{j \neq i} p_{E_j} \in \mathbf{p}'(E_j), \forall p_{E_i|A} \in \mathbf{p}'(E_i|A), \\ \exists p_{A|E_i} \in \mathbf{p}'(A|E_i), \exists p_{E_i} \in \mathbf{p}'(E_i), \\ p_{E_i|A} = \frac{p_{A|E_i} p_{E_i}}{\sum_{j=1}^{n} p_{A|E_j} p_{E_j}} \end{aligned} \tag{36}$$

Notice that because both $\mathbf{p}(A|E_i)$ and $\text{dual}\mathbf{p}(A|E_i)$ occur in Eq.(29), the associated logic interpretation about $\mathbf{p}(A|E_i)$ is always existential. This indicates that the completeness of the posterior probability $\mathbf{p}(E_i|A)$ cannot be checked by the interpretation itself. Yet the soundness of the posterior probability estimation can be checked by some interpretations such as the one in Eq.(36).

5. Concluding Remarks

In this paper, we presented a new form of imprecise probability based on generalized intervals. Generalized intervals allow the coexistence of proper and proper intervals. This enables the algebraic closure of arithmetic operations. We differentiate focal events from non-focal events by the modalities and semantics of interval probabilities. An event is focal when the semantics associated with its interval probability is universal, whereas it is non-focal when the semantics is existential. This differentiation allows us to have a simple and unified representation based on a logic coherence constraint, which is a stronger restriction than the regular 2-monotoniciy. This stronger requirement appears to be the cost we pay for the algebraic closure.

New rules of conditioning and updating are defined with generalized intervals. The new conditional probabilities ensure the algebraic relation with marginal interval probabilities. It is also shown that the new Bayes' updating rule is a generalization of the 2-monotone tight envelope updating rule under the new representation. This enables sequence-independent updating. Generalized intervals also allow us to interpret the algebraic relations among intervals in terms of the first-order logic. This helps us to understand the relationship between individual measurements and probability sets as well as to check completeness and soundness of bounds.

In summary, the algebraic closure of the new form provides some advantages for a simpler probability calculus, which is helpful in engineering and computer science practices. Future work may include the study of interpretation with the new form for assessment guidance. That is, we need to understand the algebraic conclusions better and take appropriate actions. Even though the computation is simplified, the completeness of lower and upper envelope estimations based on generalized intervals is not clear in general. We need to study how generalized intervals may underestimate envelopes. We also need to investigate the difference between the new and the traditional interval forms because of the logic coherence constraint.

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