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An Extended Kalman Filtering Mechanism Based on Generalized Interval Probability

Kalman filter has been widely applied for state identification in controllable systems. As a special case of the hidden Markov model, it is based on the assumption of linear dependency relationships and Gaussian noise. The classical Kalman filter does not differentiate systematic error from random error associated with observations. In this paper, we propose an extended Kalman filtering mechanism based on generalized interval probability, where state and observable variables are random intervals, and interval-valued Gaussian distributions model the noises. The prediction and update procedures in the new mechanism are derived. Two examples are used to illustrate the developed mechanism. It is shown that the method is an efficient alternative to sensitivity analysis for assessing the effect of systematic error. [DOI: 10.1115/1.4030465]

Keywords: generalized interval, generalized interval probability, Kalman filter, systematic error, manufacturing

1 Introduction

The Kalman filter [1] is a widely applied approach to analyze time series problems. It iteratively produces a statistical estimate of the underlying states in a dynamic system from a sequence of inaccurate observable data. In the classical Kalman filter model, linear relationships are assumed to exist between hidden state variables and the observable, as well as the time-dependency between state variables, subject to Gaussian noise. Some extensions have been proposed to relax these assumptions. For example, nonlinear filters [2,3] deal with nonlinear system functions with the linear approximation in calculation. The Kalman–Bucy filter [4] extends the discrete-time sequence to the continuous-time domain. The fixed interval smoother [5–7] provides an optimum estimate of state variables using a fixed interval. The set-valued Kalman filter [8,9] has the filtering mechanism to handle uncertain initial values.

The classical Kalman filter and its extensions have been applied to solving control problems, such as position localization [10–13], robot control [14–16], predictive control [17–19], and others [20]. Additionally, they have also been used in other areas. For example, they were used in navigation systems in combination with fuzzy logic [21], tracking human body motion with quaternion input [22], performance and fault diagnostics in combination with neural network [23,24], fault detection for a hydraulic actuator circuit [25], and macroeconomics [26,27].

In the aforementioned models, it is assumed that the full knowledge of noise in the stochastic state evolutions and observations is available, which follow Gaussian distributions with precise parameters. With this assumption, different sources of uncertainties and errors, particularly systematic error versus random error, are not considered. Random error, also known as irreducible uncertainty and variability, is caused by unpredictable changes and random fluctuations in the measuring instrument or external environment. It is inherently irreducible. Systematic error, also known as reducible uncertainty and incertitude, is the unknown bias introduced by an instrument or a human being. It is due to the lack of perfect knowledge. Systematic error is always associated with experimental data. It can be reduced or eliminated if its cause is identified. A common method to reduce systematic error is through a careful calibration of the measurement instruments. The classical Kalman filter does not explicitly differentiate systematic error from random error. The probability distributions used in the model only capture the compounded effect of these two. The individual effect of reducing systematic error is not easily obtainable unless sensitivity analysis is conducted.

In order to take systematic error into separate consideration, in this paper, an extended Kalman filtering mechanism based on generalized interval probability [28] is proposed. Generalized interval probability is a combination of probability and a generalized interval to quantify reducible and irreducible uncertainties simultaneously but with different forms. Here, interval values represent systematic error, whereas probability measures capture random error. Systematic error is typically estimated by the lower and upper limits of instrument accuracy (instead of probability distribution). Therefore, any measured quantity is inherently an interval value, typically with the form of nominal value $\pm$ accuracy. Without assuming any distribution between the lower and upper bounds, an interval uses the least assumption in quantifying systematic error because of the lack of knowledge.

The proposed mechanism is different from the Kalman filter extensions mentioned previously. The most similar one is the set-valued Kalman filter [8,9]. In this model, the initial value of the system state is unknown but bounded within some region. The system under uncertainty then evolves as a set of possible trajectories, which are estimated by sampling the initial state value in the region in combination with real-valued covariance. In our method, state variables directly take the values of *random intervals* in system evolution, instead of random numbers. In addition, the probability distributions associated with state variables in the set-valued Kalman filter are all Gaussian. In our mechanism, they do not have to be. The interval probability has both mean and variance as intervals. The variations associated with the mean and variance are

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independent. The corresponding interval cumulative distribution function (CDF) is regarded as a set of CDFs enclosed by the interval bounds. The actual profile of the enclosed CDF could have any shape other than Gaussian as long as it is enclosed by the interval bounds. In other words, non-Gaussian processes have been considered in our interval probability approach.

Another uniqueness of our mechanism is the application of a generalized interval [29–31]. Generalized interval is an algebraic extension of the classical interval. The classical interval is defined as a set of real numbers with lower and upper bounds. That is, $[\![\underline{x}, \bar{x}]\!] := \{x \in \mathbb{R} | \underline{x} \le x \le \bar{x}\}$. However, instead of a group, classical intervals form a semi-group based on the interval arithmetic, which is designed to generate rigorous lower and upper bounds for the worst cases. Thus, the algebraic property of interval arithmetic is not quite intuitive. For instance, to find the solution $\mathbf{x}$ of equation $[\![1{,}100]\!] + \mathbf{x} = [\![2{,}102]\!]$, $\mathbf{x} = [\![2{,}102]\!] - [\![1{,}100]\!] = [\![-98{,}101]\!]$ based on classical intervals. However, $\mathbf{x} = [\![-98{,}101]\!]$ is not the algebraic solution of the original equation. When plugged back into the equation, it leads to $[\![1{,}100]\!] + [\![-98{,}101]\!] = [\![-97{,}201]\!] \neq [\![2{,}102]\!]$. Subtraction is not the inverse operation of addition.

In contrast, a generalized interval is defined as a pair of numbers $\mathbf{x} := [\underline{x}, \bar{x}] \in \mathbb{KR}$, which is no longer restricted to the ordered bounds as $\underline{x} \le \bar{x}$. For example, both [1,2] (which is called *proper*) and [2,1] (which is called *improper*) are valid generalized intervals. By introducing the operation dual, generalized intervals form a group, because $\mathbf{x} - \mathrm{dual}(\mathbf{x}) = [\underline{x} - \underline{x}, \bar{x} - \bar{x}] = 0$, where $\mathrm{dual}[\underline{x}, \bar{x}] := [\bar{x}, \underline{x}]$. By applying operator dual and Kaucher arithmetic [32] in the calculation of the previous example, we receive $\mathbf{x} = [2{,}102] - \mathrm{dual}[1{,}100] = [1, 2]$, which is the algebraic solution, as $[\![1{,}100]\!] + [\![1, 2]\!] = [\![2{,}102]\!]$. Therefore, when solving inverse problems, classical interval arithmetic tends to overestimate the ranges. A generalized interval does not have this issue with its unique algebraic property.

In general, variables in classical interval arithmetic are treated as independent ones, which causes overestimation of interval ranges. For instance, $\mathbf{x} - \mathbf{x} = [\![\underline{x}, \bar{x}]\!] - [\![\underline{x}, \bar{x}]\!] = [\![\underline{x} - \bar{x}, \bar{x} - \underline{x}]\!] \neq 0$ when $\underline{x} \neq \bar{x}$. Here, the computation of the worst case in this equation is unnecessary. In engineering applications, when interval models and interval arithmetic are adopted for uncertainty analysis, the rigorous bounds of the system outputs are desirable. However, because of the nature of interval arithmetic, the rigorous bounds provided by the classical intervals for the worst case are too wide to be useful in many applications, which makes many practitioners hesitant to apply interval analysis in their domains.

Two types of interval range estimations need to be differentiated. One is *completeness*, which is the focus of the classical interval, and the other is *soundness*. A *complete* interval estimation encloses all possible solutions between the lower and upper bounds without underestimation (as the worst-case scenario), whereas all solutions enclosed in a *sound* interval estimation are true ones without any overestimation. Complete and sound estimations are also known as outer and inner estimations, respectively. Classical interval arithmetic was developed to provide complete estimation, whereas Kaucher arithmetic in a generalized interval gives sound estimation if improper intervals are applied. An ideal situation is that an interval estimation is both complete and sound. Unfortunately, classical interval arithmetic guarantees completeness at the cost of soundness. It pessimistically overestimates the true solution. Techniques have been developed to alleviate the overestimation problem, such as subpaving and Taylor series high-order enclosure. However, they require costly computation in order to improve the soundness of estimation. In contrast, a generalized interval is more efficient in providing sound estimations [31].

A generalized interval based on Kaucher arithmetic does not always provide complete estimations. However, it can provide sound estimates relatively efficiently. For instance, in the previous example of solving inverse problem $A + X = B$, the sound estimation of X can be easily obtained as $X = B - \mathrm{dual}A$, which is not trivial if the classical interval is used. In many applications, sound estimation of output variations as a result of uncertain inputs provides more information than complete but overly pessimistic bounds. Thus, methods such as sensitivity analysis are more acceptable in these applications than rigorous bounds in interval analysis. Although the complete range estimation of possible outputs with respect to inputs cannot be obtained using local sensitivity analysis, the analysis results still provide useful information to assess the effect of uncertain inputs. The generalized interval approach can be regarded as an efficient alternative to traditional sensitivity analysis that has the property of soundness without overestimation.

In addition, generalized interval provides more semantic capabilities than classical interval with its modality (proper or improper). It can express the design intent of controllable and uncontrollable for the interval variables by combining quantifiers $\forall$ and $\exists$. Both generalized interval and classical interval describe a range of variations with the interval width indicating the degree of uncertainty. However, a generalized interval can also specify whether a variable is controllable or uncontrollable by applying the modality. The assignment of a proper interval to an input variable, which has the same format as the classical interval, implies that the value variation of this variable is not controllable. That is, the variable can take any value within its corresponding interval range and is out of our control. In contrast, the assignment of an improper interval as the input implies that the variable's value can be selected by us to compensate the uncertainty associated with other uncontrollable variables such that the final uncertainty of the output can be reduced or able to meet a target range. Controllable variables modeled by improper intervals have the physical meanings of "choice," "selected," "flexible," and "adjustable" in the application of control, whereas uncontrollable variables modeled by proper intervals have the meanings of "imposed," "assigned," "rigid," and "unadjustable" (see Refs. [29–31] for details of the unique algebraic and semantic properties of a generalized interval).

As an illustrative example, a generalized interval can be applied to capture the process-oriented semantics in manufacturing. A tolerance chain of dimensions is specified as $L_a + L_b = L_c$ for an assembly of parts A and B, which have the respective dimensions of $L_a = [50.78, 50.81]$ and $L_b = [11.42, 11.43]$. When parts A and B are supplied by suppliers, L_a and L_b are uncontrollable in the assembly line. The dimensions of the supplied parts can take any possible values, respectively, from L_a and L_b. The resulting $L_c = [62.20, 62.24]$ is a complete estimation that considers all possible values of $L_a + L_b$. The tolerance chain $L_a + L_b = L_c$ with proper L_a, L_b, and L_c has the following interpretation: for all possible values within $[\![50.78, 50.81]\!]$ and $[\![11.42, 11.43]\!]$, the dimension of final assembly must be in the range of $[\![62.20, 62.24]\!]$. When part B is built in-house, the tolerance chain can be reformulated as controllable with $L_b = [11.43, 11.42]$ being an improper interval. The manufacturer has the flexibility to select one out of many in-house built parts to fit into the assembly in order to meet a specific target of overall tolerance. This process is known as selective assembly. In other words, even though the tolerance of fabricated part B remains the same as before, the selective assembly process allows us to select individual parts with some degree of control so that the accumulated tolerance is within the limit of $L_c = [62.21, 62.23]$. A different scenario is that part A is rigid, whereas part B is made of flexible or compliant materials. As a result, the dimension of part B becomes easily adjustable to fit into the target overall tolerance of L_c. In either scenario, the uncertainty associated with the assembly can be reduced by controlling the dimension of part B. The tolerance chain with proper L_a, improper L_b, and proper L_c has the following interpretation: for all possible values in $[\![50.78, 50.81]\!]$, there exists a value in $[\![11.42, 11.43]\!]$, such that the dimension of final assembly meets the target range of $[\![62.21, 62.23]\!]$. When a variable is controllable, its value can be selected from a range. However, when it is uncontrollable, its value could be any within the range. Therefore, at the syntax level, a generalized interval is a pair of numbers which specify the range of variation. At the semantics level, it describes the controllability.

An improper interval as an input of a system implies that its value can be selected or adjusted in order to compensate the uncertainty associated with other uncontrollable variables, such that the final uncertainty of output can be controlled within a given range.

In this paper, we propose an extended Kalman filtering mechanism based on generalized interval probability, where state and observable variables take the values of random intervals. The prediction and update procedures of this new mechanism are derived. With simple computation, the system dynamics subject to the uncertainties associated with initial states, probability distributions, and experimental measurement can be modeled. Therefore, this approach is an efficient alternative to sensitivity analysis. The new approach is demonstrated by two examples. One is a simple example of random constant estimation, and the other is dimensional variation propagation in a multistage assembly line. The dimension variation of a part or an assembly is its deviation from the specified value in design. The deviation is due to the imperfection of the manufacturing and assembly processes. Typically, the assembly line is monitored by sensors in order to control the quality of product. Yet, the imperfection of sensors also introduces uncertainty in the control process. Here, a Kalman filter is used to model the dimensional variation propagation in a multistage assembly line, where the product flows through multiple stations in sequence.

The original contributions of the proposed method include the simultaneous quantification of both systematic and random errors in the Kalman filter. With the consideration of sensing and measurement bias, the prediction of system states is more robust. Non-Gaussian process can be incorporated in modeling the system dynamics. In addition, the employment of a generalized interval as an extension of the classical interval with the differentiation between controllable and uncontrollable variables enables fine-grained models of process control. The good algebraic property of the generalized interval renders our mechanism an efficient alternative to traditional sensitivity analysis.

In the remainder of the paper, we first give some background information pertinent to the Kalman filter, interval probability, generalized interval, and generalized interval probability in Sec. 2. The extended Kalman filtering mechanism with prediction and update steps is described in Sec. 3. The proposed mechanism is demonstrated with two examples in Sec. 4.

2 Background

2.1 Kalman Filter. A linear discrete-time dynamic system is described as

$$X_k = A_k X_{k-1} + B_k U_k + w_k \tag{1}$$

$$Y_k = C_k X_k + v_k \tag{2}$$

where $X_k \in \mathbb{R}^n$ is the state variable at time k, $A_k \in \mathbb{R}^{n\times n}$ is the state matrix, $B_k \in \mathbb{R}^{n\times p}$ is the control input matrix, $U_k \in \mathbb{R}^p$ is the control input, $C_k \in \mathbb{R}^{m\times n}$ is the observation matrix, $Y_k \in \mathbb{R}^m$ is the observation, $w_k \in \mathbb{R}^n$ is the system disturbance, and $v_k \in \mathbb{R}^m$ is the measurement error. Equation (1) is called the state equation, and Eq. (2) is called the observation equation.

With the assumption of Gaussian distribution, the first- and second-order statistical moments of X_k are used to characterize its randomness, which are the mean state vector $X_k^* = \mathbb{E}\{X_k\}$ and covariance matrix $\mathrm{Var}(X_k) = \mathbb{E}\{(X_k - X_k^*)(X_k - X_k^*)^{\mathrm{T}}\}$. w_k and v_k are defined as white noise with zero mean values and covariances of $\mathrm{Var}(w_k)$ and $\mathrm{Var}(v_k)$, respectively. They are uncorrelated with the initial state vector X_0, which is characterized by X_0^* and $\mathrm{Var}(X_0)$.

The distribution of X_k for this system can be determined by the classical Kalman filter with two steps. In the *prediction* step, the mean and variance of the state vector at time k are calculated as

$$\hat{X}_k^- = A_k \hat{X}_{k-1} + B_k U_k$$
$$\mathrm{Var}(\hat{X}_k^-) = A_k \mathrm{Var}(\hat{X}_{k-1}) A_k^{\mathrm{T}} + \mathrm{Var}(w_k) \tag{3}$$

and are updated during the *update* step as

$$\hat{X}_k = \hat{X}_k^- + K_k[Y_k - C_k \hat{X}_k^-]$$
$$\mathrm{Var}(\hat{X}_k) = \mathrm{Var}(\hat{X}_k^-) - K_k C_k \mathrm{Var}(\hat{X}_k^-) \tag{4}$$

where $-$ denotes the predicted state value based on the previous one, $\hat{}$ denotes the updated state value given the current observation, and $K_k = \mathrm{Var}(\hat{X}_k^-) C_k^{\mathrm{T}} [C_k \mathrm{Var}(\hat{X}_k^-) C_k^{\mathrm{T}} + \mathrm{Var}(v_k)]^{-1}$ is the so-called Kalman gain, which reaches the optimum with the minimum error covariance.

2.2 Interval Probability. Interval probability or imprecise probability quantify variability and incertitude simultaneously. Many representations of imprecise probability have been developed. For example, the Dempster–Shafer evidence theory [33,34] uses a belief–plausibility pair to characterize the possibilities in support of and against a hypothesis. The coherent lower prevision theory [35] models the belief and disbelief via the lower and upper previsions. The interval probability arithmetic [36] was proposed to compute the lower and upper bounds for a range of probability measures by applying interval arithmetic. Probability bound analysis [37,38] specifies interval probability as a pair of cumulative distribution functions for uncertain variables. F-probability [39] models interval probability with a set of classical probability measures that satisfy the three axioms of Kolmogorov. A cloud [40] is an interval-valued membership function in a fuzzy set that specifies the probability measures. Generalized interval probability [41,42] uses a generalized interval [43] to define imprecise probability that has a similar algebraic structure as the classical probability.

The samples drawn from an interval probability distribution are random intervals. There are several ways to define and compute the mean and variance for random intervals. Körner [44] found that the variance of a random compact convex set is the sum of the variance of its Steiner point and the variance of a centered random set via an L_2 metric in the corresponding space of the support functions. Thus, the variance of the interval data is obtained by averaging the sum of the variances of its lower and upper bounds. Based on the variance definition in [44], Sun [45] proposed a hierarchical normal model to represent random intervals and estimate the mean and variance parametrically. Other researchers treat the computation of the variance of random intervals as an optimization problem, and the variance of random intervals specifies the lower and upper bounds of the variance for the random values sampled from the random intervals. Therefore, the lower and upper bounds of variance can be computed separately. Ferson et al. [37] proposed an approach to compute the lower bound of variance in a quadratic time, whereas the upper bound is much more complex in an NP-hard time. Gang et al. [46] showed that the computing of lower and upper bounds of variance in some practical case can be both finished in linear time by applying some constraint techniques.

2.3 Generalized Interval. A generalized interval [29,30,32,43,47] is an algebraic and semantic extension of the classical interval [48]. An operator Δ maps a generalized interval $\mathbf{x}$ to a classical interval, defined as

$$[\underline{x}, \bar{x}]^{\Delta} = \begin{cases} [\![\underline{x}, \bar{x}]\!], & \text{if } \underline{x} \le \bar{x} \\ [\![\bar{x}, \underline{x}]\!], & \text{if } \underline{x} \ge \bar{x} \end{cases} \tag{5}$$

$\mathbf{x}$ is *proper* when $\underline{x} \le \bar{x}$ and denoted as $x \in \mathbb{IR}$. $\mathbf{x}$ is *improper* when $\underline{x} \ge \bar{x}$ and denoted as $x \in \overline{\mathbb{IR}}$. The pointwise interval $\mathbf{x}$ with $\underline{x} = \bar{x}$ can be either proper or improper. The property of proper or improper is referred to as the *modality* of the interval. Operators *pro*

and *imp* return proper and improper intervals, respectively, and are defined as

$$\text{pro:} = [\min(\underline{x}, \bar{x}), \max(\underline{x}, \bar{x})] \tag{6}$$

$$\text{imp:} = [\max(\underline{x}, \bar{x}), \min(\underline{x}, \bar{x})] \tag{7}$$

The relationship between proper and improper intervals is established by an operator dual, defined as $\text{dual}([\underline{x}, \bar{x}]) := [\bar{x}, \underline{x}]$. Functions inf and sup return the lower and upper bounds of $\mathbf{x}$, respectively, i.e., $\mathbf{x}.\inf = \underline{x}$ and $\mathbf{x}.\sup = \bar{x}$. For two generalized intervals $\mathbf{x} = [\underline{x}, \bar{x}]$ and $\mathbf{y} = [\underline{y}, \bar{y}]$, $\mathbf{x} = \mathbf{y}$ if and only if $\underline{x} = \underline{y}$ and $\bar{x} = \bar{y}$.

The arithmetic for generalized interval is called Kaucher arithmetic [32], which coincides with the classical interval arithmetic when only proper intervals are involved. Inclusion monotonicity is a property of Kaucher arithmetic and is expressed as

$$\mathbf{y}_1 \subseteq \mathbf{x}_1, \qquad \mathbf{y}_2 \subseteq \mathbf{x}_2 \Rightarrow \mathbf{y}_1 \cdot \mathbf{y}_2 \subseteq \mathbf{x}_1 \cdot \mathbf{x}_2 \tag{8}$$

where $\cdot \in \{+, -, \times, \div\}$, $\mathbf{x}_1$, $\mathbf{y}_1$, $\mathbf{x}_2$, and $\mathbf{y}_2$ are generalized intervals. It also states that if $F(\mathbf{y}_1, \ldots, \mathbf{y}_n)$ is an interval extension of function f which only involves the operations $\cdot \in \{+, -, \times, \div\}$, then we have

$$F(\mathbf{y}_1, \ldots, \mathbf{y}_n) \subseteq F(\mathbf{x}_1, \ldots, \mathbf{x}_n) \tag{9}$$

if $\mathbf{y}_i \subseteq \mathbf{x}_i$ for $i = 1, \ldots, n$.

2.4 Generalized Interval Probability. Generalized interval probability [42,49] is defined as $\mathbf{p}: \mathcal{A} \to 0, 1 \times 0, 1$ given a sample space Ω and a σ-algebra $\mathcal{A}$ of random events over Ω, which obeys the Kolmogorov axioms: (1) $\mathbf{p}(\Omega) = [1, 1]$; (2) $[0, 0] \leq \mathbf{p}(E) \leq [1, 1] (\forall\, E \in \mathcal{A})$; and (3) for any countable mutually disjoint events $E_i \cap E_j = \Phi (i \neq j)$, $\mathbf{p}(\cup_{i=1}^{n} E_i) = \sum_{i=1}^{n} \mathbf{p}(E_i)$. An interval-valued probability $\mathbf{p} = [\underline{p}, \bar{p}] \in \mathbb{KR}$ is a generalized interval without the restriction of $\underline{p} \leq \bar{p}$, and it implies $\mathbf{p}(\phi) = [0, 0]$.

The interval-valued probability distribution function can be represented in a similar way with the traditional one but with interval-valued mean and variance. For example, an interval-valued Gaussian distribution can be denoted as $\mathbb{N}(\mathbf{x}^*, \mathbf{Var}(\mathbf{x}))$, where $\mathbf{x}^*$ is the mean value and $\mathbf{Var}(\mathbf{x})$ is the variance.

Figure 1 shows four cases of interval-valued Gaussian distributions. Figure 1(*a*) shows the CDF when the parameters are precise (i.e., point-wise interval parameters), which is equivalent to the traditional Gaussian distribution. The CDFs with either interval-valued mean or variance are shown in Figs. 1(*b*) and 1(*c*), respectively, in which the curve with a dashed line is the precise one originally in Fig. 1(*a*) and are bounded by the interval CDF. The most general case is the one with both interval-valued mean and variance as in Fig. 1(*d*). Its lower and upper CDFs, indicated as the curves with solid lines, are piecewise functions, described by

$$F(\bar{x}) = \begin{cases} \dfrac{1}{2}\left(1 + \text{erf}\left(\dfrac{x - \underline{x}^*}{\sqrt{2}\text{Var}(\bar{x})}\right)\right), & \text{when } x \leq \underline{x}^* \\ \dfrac{1}{2}\left(1 + \text{erf}\left(\dfrac{x - \underline{x}^*}{\sqrt{2}\text{Var}(\underline{x})}\right)\right), & \text{when } x > \underline{x}^* \end{cases}$$

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where $\underline{x}^*$ and $\bar{x}^*$ are the lower and upper bounds of $\mathbf{x}^*$; $\text{Var}(\underline{x})$ and $\text{Var}(\bar{x})$ are the lower and upper bounds of $\mathbf{Var}(\mathbf{x})$; and the error function is given as

$$\text{erf}(x) = 2/\sqrt{\pi} \int_0^x \text{e}^{-t^2}\, \text{d}t$$

Note that the lower and upper CDFs of an interval Gaussian distribution enclose a set of distributions. These distributions are not necessarily Gaussian. Any distribution with arbitrary CDF is thus considered as long as the CDF is enclosed by the interval CDF. When the distribution type for an interval random variable is not specified, we only consider its mean and variance and denote it as $\mathbf{x} \sim \mathbb{N}(\mathbf{x}^*, \mathbf{Var}(\mathbf{x}))$. Therefore, the notation $\mathbb{N}(\mathbf{x}^*, \mathbf{Var}(\mathbf{x}))$ used here does not assume a Gaussian distribution in the physical process. Rather, a set of possible distributions that are enclosed by the two Gaussian bounds are modeled.

Given a statistical distribution with interval parameters, the inverse transform can be used to calculate the lower and upper bounds of the interval random variables. For the distributions in Fig. 1, given a probability value $F \in [0, 1]$, the lower and upper bounds of an interval random variable are calculated as

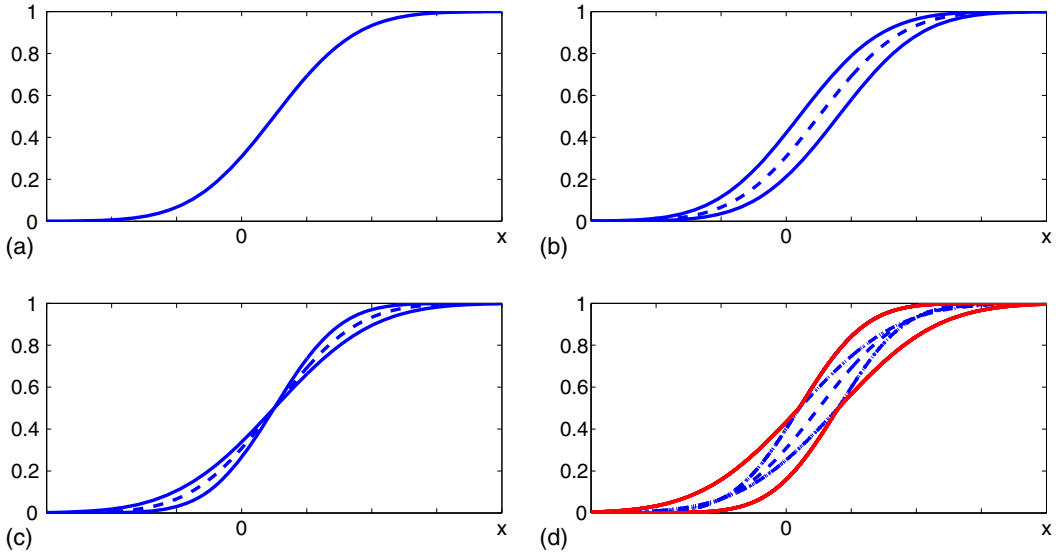

Fig. 1 Cumulative distribution functions of Gaussian distribution with precise and imprecise parameters: (*a*) *x*~(0.5,1), (*b*) *x*~([0.2,0.8],1), (*c*) *x*~(0.5,[0.8,1.2]), and (*d*) *x*~([0.2,0.8],[0.8,1.2])

$$\underline{x}_i = \underline{x}^* + \sqrt{2}\mathrm{Var}(x_l)\mathrm{erf}^{-1}(2F-1)$$
$$\bar{x}_i = \bar{x}^* + \sqrt{2}\mathrm{Var}(x_u)\mathrm{erf}^{-1}(2F-1)(\forall\, i \in [1,n]) \quad (11)$$

where

$$\mathrm{Var}(x_l) = \begin{cases} \mathrm{Var}(\underline{x}), i \le n/2 \\ \mathrm{Var}(\bar{x}), i > n/2 \end{cases}, \qquad \mathrm{Var}(x_u) = \begin{cases} \mathrm{Var}(\bar{x}), i \le n/2 \\ \mathrm{Var}(\underline{x}), i > n/2 \end{cases}$$

3 Extended Kalman Filter Based on Generalized Interval Probability

In the proposed extended Kalman filter, measurement error $\mathbf{v}_k \in \mathbb{KR}^m$ and system disturbance $\mathbf{w}_k \in \mathbb{KR}^n$ are described as interval-valued probability distributions, which capture both systematic and random errors. $\mathbf{U}_k \in \mathbb{KR}^n$ is an interval-valued vector that defines the lower and upper limits of controllable inputs. $\mathbf{X}_k \in \mathbb{KR}^n$ and $\mathbf{Y}_k \in \mathbb{KR}^m$ are interval-valued vectors for the state variable and observable, respectively. Then, the state-space model in Eqs. (2) and (3) can be reformulated as

$$\mathbf{X}_k = A_k\mathbf{X}_{k-1} + B_k\mathbf{U}_k + \mathbf{w}_k \quad (12)$$

$$\mathbf{Y}_k = C_k\mathbf{X}_k + \mathbf{v}_k \quad (13)$$

where Kaucher arithmetic [32] is applied in the calculation. Before the state vector in interval-valued state space model (Eqs. (12) and (13)) is estimated, several basic definitions for the interval-valued probability must be defined first.

DEFINITION 1. *The expected value for a set of random intervals* $x \in \mathbb{KR}$ *is defined as* $\mathbf{x}^* = \mathbb{E}(\mathbf{x}) := [\mathbb{E}(\underline{x}), \mathbb{E}(\bar{x})] = \left[\frac{1}{n}\sum_{i=1}^n \underline{x}_i, \frac{1}{n}\sum_{i=1}^n \bar{x}_i\right]$.

DEFINITION 2. *Suppose a random interval* $\mathbf{x}$ *with expected value* $\mathbf{x}^*$*; the variance of* $\mathbf{x}$ *is defined as a* 2×1 *vector, which is*

$$\mathbf{Var}(\mathbf{x}) = \begin{pmatrix} \mathbf{Var}_{\mathbf{xx}}(\mathbf{x}) \\ \mathbf{Var}_{\mathbf{x}\mathrm{dual}\mathbf{x}}(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \mathbb{E}[(\mathbf{x}-\mathrm{dual}\mathbf{x}^*)(\mathbf{x}-\mathrm{dual}\mathbf{x}^*)] \\ \mathbb{E}[(\mathbf{x}-\mathrm{dual}\mathbf{x}^*)(\mathrm{dual}\mathbf{x}-\mathbf{x}^*)] \end{pmatrix} \quad (14)$$

THEOREM 1. $\mathbf{Var}_{\mathbf{xx}}(\mathbf{x}) = [Var(\underline{x}), Var(\bar{x})]$, *i.e., it measures the respective variances of lower and upper bounds of* $\boldsymbol{x}$.

Proof. Because $\mathbf{x}^* = \mathbb{E}(\mathbf{x})$, $\mathbb{E}[(\mathbf{x}-\mathrm{dual}\mathbf{x}^*)^2] = \mathbb{E}\{[(\underline{x}-\mathbb{E}(\underline{x})), (\bar{x}-\mathbb{E}(\bar{x}))]^2\} = [\mathbb{E}(\underline{x}-\mathbb{E}(\underline{x}))^2, \mathbb{E}(\bar{x}-\mathbb{E}(\bar{x}))^2]$.
Thus, $\mathbf{Var}_{\mathbf{xx}}(\mathbf{x}) = [\mathrm{Var}(\underline{x}), \mathrm{Var}(\bar{x})]$.

THEOREM 2. $\mathbf{Var}_{\mathbf{x}\mathrm{dual}\mathbf{x}}(\mathbf{x}) = Cov(\underline{x}, \bar{x})$, *i.e., it measures the covariance between the lower and upper bounds of* $\boldsymbol{x}$.

Proof. Because $\mathbf{x}-\mathrm{dual}\mathbf{x}^* = [\underline{x}-\mathbb{E}(\underline{x}), \bar{x}-\mathbb{E}(\bar{x})]$, $\mathrm{dual}\mathbf{x}-\mathbf{x}^* = \mathrm{dual}(\mathbf{x}-\mathrm{dual}\mathbf{x}^*)$, $\mathbb{E}\{(\mathbf{x}-\mathrm{dual}\mathbf{x}^*)(\mathrm{dual}\mathbf{x}-\mathbf{x}^*)\} = \mathbb{E}\{[\underline{x}-\mathbb{E}(\underline{x}), \bar{x}-\mathbb{E}(\bar{x})][\bar{x}-\mathbb{E}(\bar{x}), \underline{x}-\mathbb{E}(\underline{x})]\}$. Considering with measurement error in practice, random interval-valued data $\mathbf{x}$ measured by the same tool should have the same systematic error when human bias is ignored. Thus, there only exists $\mathbf{x} \geqslant \mathbf{x}^*$ or $\mathbf{x} \leqslant \mathbf{x}^*$, and $0 \notin [\underline{x}-\mathbb{E}(\underline{x}), \bar{x}-\mathbb{E}(\bar{x})]$. Then $\mathbb{E}\{[\underline{x}-\mathbb{E}(\underline{x}), \bar{x}-\mathbb{E}(\bar{x})][\bar{x}-\mathbb{E}(\bar{x}), \underline{x}-\mathbb{E}(\underline{x})]\} = [\mathbb{E}\{(\underline{x}-\mathbb{E}(\underline{x}))(\bar{x}-\mathbb{E}(\bar{x}))\}, \mathbb{E}\{(\underline{x}-\mathbb{E}(\underline{x}))(\bar{x}-\mathbb{E}(\bar{x}))\}]$ based on Kaucher arithmetic. Therefore, $\mathbf{Var}_{\mathbf{x}\mathrm{dual}\mathbf{x}}(\mathbf{x}) = \mathrm{Cov}(\underline{x}, \bar{x})$.

COROLLARY 1. $\mathbf{Var}_{\mathbf{x}\mathrm{dual}\mathbf{x}}(\mathbf{x}) \subseteq \mathbf{Var}_{\mathbf{xx}}(\mathbf{x})$ *when* $\mathbf{Var}_{\mathbf{xx}}(\mathbf{x}) \in \mathbb{IR}$.

Proof. When $\mathbf{Var}_{\mathbf{xx}}(\mathbf{x}) \in \mathbb{IR}$, $\mathbf{x}-\mathrm{dual}\mathbf{x}^*$ in $\mathbb{E}[(\mathbf{x}-\mathrm{dual}\mathbf{x}^*)^2]$ should be proper too based on Kaucher arithmetic. Then, $\mathrm{dual}\mathbf{x}-\mathbf{x}^*$ is improper, and $\mathrm{dual}\mathbf{x}-\mathbf{x}^* \subseteq \mathbf{x}-\mathrm{dual}\mathbf{x}^*$. Because of the inclusion property, $(\mathbf{x}-\mathrm{dual}\mathbf{x}^*)(\mathrm{dual}\mathbf{x}-\mathbf{x}^*) \subseteq (\mathbf{x}-\mathrm{dual}\mathbf{x}^*)^2$. Therefore, $\mathbf{Var}_{\mathbf{x}\mathrm{dual}\mathbf{x}}(\mathbf{x}) \subseteq \mathbf{Var}_{\mathbf{xx}}(\mathbf{x})$.

DEFINITION 3. *[50,28]: Let* $\mathbf{F}(x) = [\underline{F}(x), \bar{F}(x)]$ *be a generalized interval function. The first derivative of* $\mathbf{F}(x)$ *with respect to* $x \in [\underline{x}, \bar{x}]^\Delta$ *is defined as*

$$\frac{d\mathbf{F}(x)}{dx} := \lim_{\Delta x\to 0}\frac{\mathbf{F}(x+\Delta x)-\mathrm{dual}\mathbf{F}(x)}{\Delta x} = \left[\lim_{\Delta x\to 0^-}\frac{\underline{F}(x+\Delta x)-\underline{F}(x)}{\Delta x}, \lim_{\Delta x\to 0^+}\frac{\bar{F}(x+\Delta x)-\bar{F}(x)}{\Delta x}\right] \quad (15)$$

The property $d(\mathrm{dual}\mathbf{F}(x))/dx = \mathrm{dual}(d\mathbf{F}(x)/dx)$ *holds for* $d\mathbf{F}(x)/dx$.

DEFINITION 4. *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two random intervals, and* $\mathbf{x}^*$ *and* $\mathbf{y}^*$ *be their respective expected values.* $\mathbf{x}$ *and* $\mathbf{y}$ *are called uncorrelated if* $0 \in \mathbb{E}[(\mathbf{x}-\mathrm{dual}\mathbf{x}^*)(\mathbf{y}-\mathrm{dual}\mathbf{y}^*)]^\Delta$.

DEFINITION 5. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be two interval-valued matrices. If* $(\mathbf{A})_{ij} = (\mathbf{B})_{ji}$, *where* $i = 1, \ldots, n$ *and* $j = 1, \ldots, m$, $\mathbf{B}$ *is the transpose matrix of* $\mathbf{A}$, *denoted as* $\mathbf{B} = \mathbf{A}^{\mathrm{T}}$.

DEFINITION 6. *An* $n \times n$ *interval-valued matrix* $\mathbf{A}$ *is invertible if there exists an* $n \times n$ *interval-valued matrix* $\mathbf{B}$ *such that* $I_n \subseteq \mathbf{AB}$, *where* I_n *is an* $n \times n$ *identity matrix and the multiplication between* $\mathbf{A}$ *and* $\mathbf{B}$ *is based on Kaucher arithmetic.* $\mathbf{B}$ *is called the inverse of* $\mathbf{A}$, *denoted as* $\mathbf{A}^{-1}$.

The mean and variance for the interval-valued state space model in Eqs. (12) and (13) can be estimated as follows. Suppose that the initial state vector $\mathbf{X}_0$ is associated with an interval probability distribution with interval mean $\mathbf{X}_0^*$ and interval variance $\mathbf{Var}(\mathbf{X}_0)$, and so does the noise, $\mathbf{v}_k$ with zero mean and $\mathbf{Var}(\mathbf{v}_k)$, $\mathbf{w}_k$ with zero mean and $\mathbf{Var}(\mathbf{w}_k)$, where $\mathbf{Var}(\mathbf{x}_0)$, $\mathbf{Var}(\mathbf{v}_k)$, and $\mathbf{Var}(\mathbf{w}_k)$ are defined as in Definition 2, and $\mathbf{v}_k$ and $\mathbf{X}_k$ are uncorrelated as defined in Definition 4. The expected value of $\mathbf{X}_k$ based on the previous state $\mathbf{X}_{k-1}$ is computed by

$$\mathbf{X}_k^* = \mathbb{E}[A_k\mathbf{X}_{k-1} + B_k\mathbf{U}_k + \mathbf{w}_k] = A_k\mathbf{X}_{k-1}^* + B_k\mathbf{U}_k^* \quad (16)$$

Substituting Eqs. (12) and (16) into Eq. (14), the error covariance of the state vector $\mathbf{Var}(\mathbf{X}_k)$ is calculated as

$$\begin{aligned}\mathbf{Var}(\mathbf{X}_k) &= \begin{pmatrix} \mathbf{Var}_{\mathbf{xx}}(\mathbf{X}_k) \\ \mathbf{Var}_{\mathbf{x}\mathrm{dual}\mathbf{x}}(\mathbf{X}_k) \end{pmatrix} \\ &= \begin{pmatrix} \mathbb{E}[A_k(\mathbf{X}_{k-1}-\mathrm{dual}\mathbf{X}_{k-1}^*)(\mathbf{X}_{k-1}-\mathrm{dual}\mathbf{X}_{k-1}^*)^{\mathrm{T}}A_k^{\mathrm{T}}] \\ \mathbb{E}[A_k(\mathbf{X}_{k-1}-\mathrm{dual}\mathbf{X}_{k-1}^*)(\mathrm{dual}\mathbf{X}_{k-1}-\mathbf{X}_{k-1}^*)^{\mathrm{T}}A_k^{\mathrm{T}}] \end{pmatrix} \\ &\quad + \begin{pmatrix} \mathbb{E}[\mathbf{w}_k\mathbf{w}_k^{\mathrm{T}}] \\ \mathbb{E}[\mathbf{w}_k\mathrm{dual}\mathbf{w}_k^{\mathrm{T}}] \end{pmatrix} \\ &= \begin{pmatrix} A_k\mathbf{Var}_{\mathbf{xx}}(\mathbf{X}_{k-1})A_k^{\mathrm{T}} + \mathbf{Var}_{\mathbf{ww}}(\mathbf{w}_k) \\ A_k\mathbf{Var}_{\mathbf{x}\mathrm{dual}\mathbf{x}}(\mathbf{X}_{k-1})A_k^{\mathrm{T}} + \mathbf{Var}_{\mathbf{w}\mathrm{dual}\mathbf{w}}(\mathbf{w}_k) \end{pmatrix} \\ &= A_k\mathbf{Var}(\mathbf{X}_{k-1})A_k^{\mathrm{T}} + \mathbf{Var}(\mathbf{w}_k)\end{aligned} \quad (17)$$

Therefore, the estimated $\mathbf{X}_k$ in the prediction step will be

$$\hat{\mathbf{X}}_k^- = A_k\hat{\mathbf{X}}_{k-1} + B_k\mathbf{U}_k$$
$$\mathbf{Var}(\hat{\mathbf{X}}_k^-) = A_k\mathbf{Var}(\hat{\mathbf{X}}_{k-1})A_k^{\mathrm{T}} + \mathbf{Var}(\mathbf{w}_k) \quad (18)$$

If the observation $\mathbf{Y}$ is described by $\mathbf{X}$ via a linear relationship in Eq. (13), $\mathbf{X}$ can be approximated by $\mathbf{Y}$ in the reverse linear relation, such as $\hat{\mathbf{X}} = \mathbf{aY} + \mathbf{b}$ [51].

The difference between the approximation $\hat{\mathbf{X}}$ and the true value $\mathbf{X}$ can be quantified by a mean squared error, defined as follows.

DEFINITION 7. *Let* $\mathbf{X}$ *be an* $n \times 1$ *interval vectors, and* $\hat{\mathbf{X}}$ *be its estimation. The estimation error is given by* $\mathbf{e} = v - \mathrm{dual}\hat{\mathbf{X}}$, *and its mean squared error is given by the trace of the error covariance matrix as*

$$\mathbf{J} = \mathrm{tr}\{\mathbb{E}\{(\mathbf{X}-\mathrm{dual}\hat{\mathbf{X}})(\mathbf{X}-\mathrm{dual}\hat{\mathbf{X}})^{\mathrm{T}}\}\} \quad (19)$$

The trace of an interval matrix $\mathbf{A}$ has the property that $\mathrm{tr}(\mathrm{dual}\mathbf{A}) = \mathrm{dual}(\mathrm{tr}(\mathbf{A}))$. As a linear map, $\mathrm{tr}(\mathbf{A}) = \mathrm{tr}(\mathbf{A}^{\mathrm{T}})$, $\mathrm{tr}(\mathbf{A}+\mathbf{B}) = \mathrm{tr}(\mathbf{A}) + \mathrm{tr}(\mathbf{B})$, and $\mathrm{tr}(\mathbf{AB}^{\mathrm{T}}) = \mathrm{tr}(\mathbf{BA}^{\mathrm{T}})$.

Substituting $\hat{\mathbf{X}} = \mathbf{aY} + \mathbf{b}$ into Eq. (19), we have

$$\begin{aligned}\mathbf{J} = \mathrm{tr}\{&\mathbb{E}\{(\mathbf{X} - \mathrm{dual}\mathbf{X}^*)(\mathbf{X} - \mathrm{dual}\mathbf{X}^*)^{\mathrm{T}}\} \\ &- \mathbb{E}\{(\mathbf{X} - \mathrm{dual}\mathbf{X}^*)(\mathrm{dual}(\mathbf{aY}) + \mathrm{dual}\mathbf{b} - \mathbf{X}^*)^{\mathrm{T}}\} \\ &- \mathbb{E}\{(\mathrm{dual}(\mathbf{aY}) + \mathrm{dual}\mathbf{b} - \mathbf{X}^*)(\mathbf{X} - \mathrm{dual}\mathbf{X}^*)^{\mathrm{T}}\} \\ &+ \mathbb{E}\{(\mathrm{dual}(\mathbf{aY}) + \mathrm{dual}\mathbf{b} - \mathbf{X}^*)(\mathrm{dual}(\mathbf{aY}) + \mathrm{dual}\mathbf{b} - \mathbf{X}^*)^{\mathrm{T}}\}\end{aligned} \tag{20}$$

in which each term will be analyzed separately. The terms from left to right at the right-hand side of Eq. (20) are denoted by T1–T4, respectively, as

$$\begin{aligned}\mathrm{T1} &= \mathrm{tr}(\mathbf{Var}_{\mathbf{xx}}(\mathbf{X})) \\ \mathrm{T2} &= \mathrm{tr}\{\mathbf{Var}_{\mathbf{x}\cdot\mathrm{dual}\mathbf{y}}(\mathbf{XY}) \cdot \mathbf{a}^{\mathrm{T}} + \mathbb{E}\{(\mathbf{X} - \mathrm{dual}\mathbf{X}^*) \\ &\quad \times (\mathbf{a} \cdot \mathrm{dual}\mathbf{Y}^* + \mathrm{dual}\mathbf{b} - \mathbf{X}^*)^{\mathrm{T}}\}\} \\ \mathrm{T3} &= \mathrm{tr}\{\mathbf{a} \cdot \mathbf{Var}_{\mathrm{dual}\mathbf{y}\cdot\mathbf{x}}(\mathbf{XY})^{\mathrm{T}} + \mathbb{E}\{(\mathbf{a} \cdot \mathrm{dual}\mathbf{Y}^* + \mathrm{dual}\mathbf{b} - \mathbf{X}^*) \\ &\quad \times (\mathbf{X} - \mathrm{dual}\mathbf{X}^*)^{\mathrm{T}}\}\} \\ \mathrm{T4} &= \mathrm{tr}\{\mathrm{dual}(\mathbf{aVar}_{\mathbf{yy}}(\mathbf{Y})\mathbf{a}^{\mathrm{T}}) + \mathbf{a} \cdot (\mathrm{dual}\mathbf{Y}^*\mathbf{Y}^{*\mathrm{T}}) \cdot \mathbf{a}^{\mathrm{T}} \\ &\quad + (\mathrm{dual}\mathbf{b} - \mathbf{X}^*)(\mathrm{dual}\mathbf{b} - \mathbf{X}^*)^{\mathrm{T}} + \mathbf{a} \cdot \mathrm{dual}\mathbf{Y}^*(\mathrm{dual}\mathbf{b} - \mathbf{X}^*)^{\mathrm{T}} \\ &\quad + (\mathrm{dual}\mathbf{b} - \mathbf{X}^*) \cdot \mathrm{dual}\mathbf{Y}^{*\mathrm{T}} \cdot \mathbf{a}^{\mathrm{T}} \\ &\quad + \mathbb{E}\{\mathrm{dual}\mathbf{a} \cdot (\mathrm{dual}\mathbf{Y} - \mathbf{Y}^*)(\mathbf{a} \cdot \mathrm{dual}\mathbf{Y}^* + \mathrm{dual}\mathbf{b} - \mathbf{X}^*)^{\mathrm{T}}\} \\ &\quad + \mathbb{E}\{(\mathbf{a} \cdot \mathrm{dual}\mathbf{Y}^* + \mathrm{dual}\mathbf{b} - \mathbf{X}^*)(\mathrm{dual}\mathbf{Y} - \mathbf{Y}^*)^{\mathrm{T}} \cdot \mathrm{dual}\mathbf{a}^{\mathrm{T}}\}\}\end{aligned} \tag{21}$$

in which $\mathrm{tr}\{\mathbf{Var}_{\mathbf{x}\mathrm{dual}\mathbf{y}}(\mathbf{XY}) \cdot \mathbf{a}^{\mathrm{T}}\} = \mathrm{tr}\{\mathbf{a}^{\mathrm{T}} \cdot (\mathbf{Var}_{\mathbf{x}\cdot\mathrm{dual}\mathbf{y}}(\mathbf{XY}))\}$, and $\mathbf{Var}_{\mathrm{dual}\mathbf{y}\cdot\mathbf{x}}(\mathbf{XY}) = \mathbf{Var}_{\mathbf{x}\mathrm{dual}\mathbf{y}}(\mathbf{XY})^{\mathrm{T}}$. Because of the linear relationship in Eq. (13), we have

$$\begin{aligned}\mathbf{Var}_{\mathbf{yy}}(\mathbf{Y}) &= \mathbb{E}[(\mathbf{Y} - \mathrm{dual}\mathbf{Y}^*)(\mathbf{Y} - \mathrm{dual}\mathbf{Y}^*)^{\mathrm{T}}] \\ &= C\mathbf{Var}_{\mathbf{xx}}(\mathbf{X})C^{\mathrm{T}} + \mathbf{Var}_{\mathbf{vv}}(\mathbf{v})\end{aligned} \tag{22}$$

and

$$\begin{aligned}\mathbf{Var}_{\mathbf{x}\mathrm{dual}\mathbf{y}}(\mathbf{XY}) &= \mathbb{E}[(\mathbf{X} - \mathrm{dual}\mathbf{X}^*)(\mathrm{dual}\mathbf{Y} - \mathbf{Y}^*)^{\mathrm{T}}] \\ &= \mathbf{Var}_{\mathbf{x}\cdot\mathrm{dual}\mathbf{x}}(\mathbf{X})C^{\mathrm{T}}\end{aligned} \tag{23}$$

Considering the properties of trace of interval matrices, and substituting Eq. (21) into Eq. (20), we have

$$\begin{aligned}\mathbf{J} = \mathrm{tr}\{&\mathbf{Var}_{\mathbf{xx}}(\mathbf{X}) + \mathrm{dual}(\mathbf{aVar}_{\mathbf{yy}}(\mathbf{Y})\mathbf{a}^{\mathrm{T}}) \\ &+ \mathbf{a} \cdot \mathrm{dual}(\mathbf{Y}^*\mathbf{Y}^{*\mathrm{T}}) \cdot \mathbf{a}^{\mathrm{T}} + (\mathrm{dual}\mathbf{b} - \mathbf{X}^*)(\mathrm{dual}\mathbf{b} - \mathbf{X}^*)^{\mathrm{T}} \\ &+ 2\mathbf{a} \cdot \mathrm{dual}\mathbf{Y}^*(\mathrm{dual}\mathbf{b} - \mathbf{X}^*)^{\mathrm{T}} - 2\mathbf{a} \cdot \mathbf{Var}_{\mathbf{x}\cdot\mathrm{dual}\mathbf{y}}(\mathbf{XY})\}\end{aligned} \tag{24}$$

where $\mathbf{Var}_{\mathbf{yy}}(\mathbf{Y})$ and $\mathbf{Var}_{\mathbf{x}\cdot\mathrm{dual}\mathbf{y}}(\mathbf{XY})$ are computed by Eqs. (22) and (23).

DEFINITION 8. *Let a generalized interval function* $\mathbf{F}$ *be continuous in the domain defined by a closed interval* $\mathbf{x}$. *The second-order derivative of* $\mathbf{F}(x) = [\underline{f}(x), \bar{f}(x)]$ *with respect to* x *is defined as* $\mathbf{F}''(x) = [\underline{f}''(x), \bar{f}''(x)]$ *if* $\underline{f}''(x)$ *and* $\bar{f}''(x)$ *exist.*

DEFINITION 9. *For a generalized interval function* $\mathbf{F}$, *if* $\mathbf{F}'(x_0) = 0$ *and* $\mathrm{pro}(\mathbf{F}''(x_0)) > 0$ *are for an* $x_0 \in \mathbf{x}^{\Delta}$, *we say* $\mathbf{F}(\mathbf{x})$ *contains the minimum values of* $\mathbf{F}$.

Here, we assume $\hat{\mathbf{X}}$ and $\mathbf{Y}$ are positively correlated, and the system bias $\mathbf{b}$ is assumed to be small such that $\mathrm{dual}\mathbf{b} - \mathbf{X}^* \geq 0$; then $\underline{J}(\mathbf{a}, \mathbf{b})$ and $\bar{J}(\mathbf{a}, \mathbf{b})$ will be

$$\begin{aligned}\underline{J}(\mathbf{a}, \mathbf{b}) = \mathrm{tr}\{&\mathbf{Var}_{\mathbf{xx}}(\mathbf{X}).\inf + \mathbf{a}.\sup \cdot \mathbf{Var}_{\mathbf{yy}}(\mathbf{Y}).\sup \cdot \mathbf{a}.\sup^{\mathrm{T}} \\ &+ \mathbf{a}.\inf \cdot \mathbf{Y}^*.\sup \cdot \mathbf{Y}^*.\sup^{\mathrm{T}} \cdot \mathbf{a}.\inf^{\mathrm{T}} \\ &+ (\mathbf{b}.\sup - \mathbf{X}^*.\sup)(\mathbf{b}.\sup - \mathbf{X}^*.\sup)^{\mathrm{T}} \\ &+ 2\mathbf{a}.\inf \cdot \mathbf{Y}^*.\sup \cdot (\mathbf{b}.\sup - \mathbf{X}^*.\sup)^{\mathrm{T}} \\ &- 2\mathbf{a}.\sup \cdot \mathbf{Var}_{\mathbf{x}\cdot\mathrm{dual}\mathbf{y}}(\mathbf{XY}).\sup\}\end{aligned} \tag{25}$$

$$\begin{aligned}\bar{J}(\mathbf{a}, \mathbf{b}) = \mathrm{tr}\{&\mathbf{Var}_{\mathbf{xx}}(\mathbf{X}).\sup + \mathbf{a}.\inf \cdot \mathbf{Var}_{\mathbf{yy}}(\mathbf{Y}).\inf \cdot \mathbf{a}.\inf^{\mathrm{T}} \\ &+ \mathbf{a}.\sup \cdot \mathbf{Y}^*.\inf \cdot \mathbf{Y}^*.\inf^{\mathrm{T}} \cdot \mathbf{a}.\sup \\ &+ (\mathbf{b}.\inf - \mathbf{X}^*.\inf)(\mathbf{b}.\inf - \mathbf{X}^*.\inf)^{\mathrm{T}} \\ &+ 2\mathbf{a}.\sup \cdot \mathbf{Y}^*.\inf \cdot (\mathbf{b}.\inf - \mathbf{X}^*.\inf)^{\mathrm{T}} \\ &- 2\mathbf{a}.\inf \cdot \mathbf{Var}_{\mathbf{x}\cdot\mathrm{dual}\mathbf{y}}(\mathbf{XY}).\inf\}\end{aligned} \tag{26}$$

Take the first-derivative of $\underline{J}(\mathbf{a}, \mathbf{b})$ and $\bar{J}(\mathbf{a}, \mathbf{b})$ in Eqs. (25) and (26) with respect to $\mathbf{b}$ and $\mathbf{a}$, respectively, and let them be equal to zero. That is,

$$\begin{aligned}&\frac{\partial \underline{J}(\mathbf{a}, \mathbf{b})}{\partial \underline{b}} = 0, \qquad \frac{\partial \bar{J}(\mathbf{a}, \mathbf{b})}{\partial \bar{b}} = 0 \\ &\frac{\partial \underline{J}(\mathbf{a}, \mathbf{b})}{\partial \bar{b}} = 2(\mathbf{b}.\sup - \mathbf{X}^*.\sup) + 2\mathbf{a}.\inf \cdot \mathbf{Y}^*.\sup = 0, \\ &\frac{\partial \bar{J}(\mathbf{a}, \mathbf{b})}{\partial \underline{b}} = 2(\mathbf{b}.\inf - \mathbf{X}^*.\inf) + 2\mathbf{a}.\sup \cdot \mathbf{Y}^*.\inf = 0 \\ &\frac{\partial \underline{J}(\mathbf{a}, \mathbf{b})}{\partial \underline{a}} = 2\mathbf{a}.\inf \cdot \mathbf{Y}^*.\sup \cdot \mathbf{Y}^*.\sup^{\mathrm{T}} \\ &\qquad + 2\mathbf{Y}^*.\sup \cdot (\mathbf{b}.\sup - \mathbf{X}^*.\sup)^{\mathrm{T}} = 0 \\ &\frac{\partial \bar{J}(\mathbf{a}, \mathbf{b})}{\partial \bar{a}} = 2\mathbf{a}.\sup \cdot \mathbf{Y}^*.\inf \cdot \mathbf{Y}^*.\inf^{\mathrm{T}} \\ &\qquad + 2\mathbf{Y}^*.\inf \cdot (\mathbf{b}.\inf - \mathbf{X}^*.\inf)^{\mathrm{T}} = 0 \\ &\frac{\partial \underline{J}(\mathbf{a}, \mathbf{b})}{\partial \bar{a}} = 2\mathbf{a}.\sup \cdot \mathbf{Var}_{\mathbf{yy}}(\mathbf{Y}).\sup - 2\mathbf{Var}_{\mathbf{x}\cdot\mathrm{dual}\mathbf{y}}(\mathbf{XY}).\sup = 0 \\ &\frac{\partial \bar{J}(\mathbf{a}, \mathbf{b})}{\partial \underline{a}} = 2\mathbf{a}.\inf \mathbf{Var}_{\mathbf{yy}}(\mathbf{Y}).\inf - 2\mathbf{Var}_{\mathbf{x}\cdot\mathrm{dual}\mathbf{y}}(\mathbf{XY}).\inf = 0\end{aligned} \tag{27}$$

which leads to

$$\begin{aligned}\mathbf{b}.\inf &= \mathbf{X}^*.\inf - \mathbf{a}.\sup \cdot \mathbf{Y}^*.\inf \\ \mathbf{b}.\sup &= \mathbf{X}^*.\sup - \mathbf{a}.\inf \cdot \mathbf{Y}^*.\sup \\ \mathbf{a}.\inf &= \mathbf{Var}_{\mathbf{x}\cdot\mathrm{dual}\mathbf{y}}(\mathbf{XY}).\inf \cdot \mathbf{Var}_{\mathbf{yy}}(\mathbf{Y}).\inf^{-1} \\ \mathbf{a}.\sup &= \mathbf{Var}_{\mathbf{x}\cdot\mathrm{dual}\mathbf{y}}(\mathbf{XY}).\sup \cdot \mathbf{Var}_{\mathbf{yy}}(\mathbf{Y}).\sup^{-1}\end{aligned} \tag{28}$$

Therefore,

$$\begin{aligned}\mathbf{b} &= \mathbf{X}^* - \mathbf{a} \cdot \mathrm{dual}\mathbf{Y}^* \\ \mathbf{a} &= \mathbf{Var}_{\mathbf{x}\cdot\mathrm{dual}\mathbf{y}}(\mathbf{XY}) \cdot \mathrm{dual}(\mathbf{Var}_{\mathbf{yy}}(\mathbf{Y})^{-1})\end{aligned} \tag{29}$$

and,

$$\hat{\mathbf{X}}(\mathbf{Y}) = \mathbf{X}^* + \mathbf{Var}_{\mathrm{dual}\mathbf{y}\cdot}(\mathbf{XY}) \cdot \mathrm{dual}(\mathbf{Var}_{\mathbf{yy}}(\mathbf{Y})^{-1})(\mathbf{Y} - \mathrm{dual}\mathbf{Y}^*) \tag{30}$$

where $\mathbf{Var}_{\mathbf{yy}}(\mathbf{Y})$ and $\mathbf{Var}_{\mathbf{x}\mathrm{dual}\mathbf{y}}(\mathbf{XY})$ are computed by Eqs. (22) and (23).

Based on Definition 2, the error covariance of $\hat{\mathbf{X}}$ estimated by $\mathbf{Y}$ is formulated as

$$\mathbf{Var}(\hat{\mathbf{X}}) = \begin{pmatrix} \mathbf{Var}_{\mathbf{xx}}(\hat{\mathbf{X}}) \\ \mathbf{Var}_{\mathbf{x}\cdot\mathrm{dual}\mathbf{x}}(\hat{\mathbf{X}}) \end{pmatrix} = \begin{pmatrix} \mathbb{E}\{(\mathbf{X}-\mathrm{dual}\hat{\mathbf{X}})(\mathbf{X}-\mathrm{dual}\hat{\mathbf{X}})^{\mathrm{T}}\} \\ \mathbb{E}\{(\mathbf{X}-\mathrm{dual}\hat{\mathbf{X}})(\mathrm{dual}\mathbf{X}-\hat{\mathbf{X}})^{\mathrm{T}}\} \end{pmatrix} \tag{31}$$

Substituting Eqs. (22), (23), and (30) into Eq. (31), we have

$$\begin{aligned} \mathbf{Var}_{\mathbf{xx}}(\hat{\mathbf{X}}) &= \mathbf{Var}_{\mathbf{xx}}(\mathbf{X}) - \mathrm{dual}(\mathbb{K}\cdot C\cdot \mathbf{Var}_{\mathbf{x}\cdot\mathrm{dual}\mathbf{x}}(\mathbf{X})) \\ &\quad - \mathbf{Var}_{\mathbf{x}\cdot\mathrm{dual}\mathbf{x}}(\mathbf{X})\cdot C^{\mathrm{T}}\cdot \mathrm{dual}\mathbb{K}^{\mathrm{T}} \\ &\quad + \mathrm{dual}\mathbb{K}\cdot C\cdot \mathrm{dual}\mathbf{Var}_{\mathbf{xx}}(\mathbf{X})\cdot C^{\mathrm{T}}\cdot \mathrm{dual}\mathbb{K}^{\mathrm{T}}\mathbf{Var}_{\mathbf{x}\mathrm{dual}\mathbf{x}}(\hat{\mathbf{X}}) \\ &= \mathbf{Var}_{\mathbf{x}\cdot\mathrm{dual}\mathbf{x}} - \mathbf{Var}_{\mathbf{xx}}(\mathbf{X})\cdot C^{\mathrm{T}}\cdot \mathbb{K}^{\mathrm{T}} \\ &\quad - \mathrm{dual}(\mathbb{K}\cdot C\cdot \mathbf{Var}_{\mathbf{xx}}(\mathbf{X})) \\ &\quad + \mathrm{dual}\mathbb{K}\cdot C\cdot \mathrm{dual}\mathbf{Var}_{\mathbf{x}\cdot\mathrm{dual}\mathbf{x}}(\mathbf{X})\cdot C^{\mathrm{T}}\cdot \mathbb{K}^{\mathrm{T}} \end{aligned} \tag{32}$$

where

$$\mathbb{K} = \mathbf{Var}_{\mathbf{x}\cdot\mathrm{dual}\mathbf{x}}(\mathbf{X})\cdot C^{\mathrm{T}}\mathrm{dual}[C\cdot \mathbf{Var}_{\mathbf{xx}}(\mathbf{X})\cdot C^{\mathrm{T}} + \mathbf{Var}_{\mathbf{vv}}(\mathbf{v})]^{-1} \tag{33}$$

The first terms in Eq. (32), $\mathbf{Var}_{\mathbf{xx}}(\mathbf{X})$ and $\mathbf{Var}_{\mathbf{x}\cdot\mathrm{dual}\mathbf{x}}(\mathbf{X})$, represent the priori covariances in Eq. (18) in the prediction step, and the rest of the terms represents the uncertainty reduction because of the measurement. Thus, combining the priori values in the prediction step with the one updated by current observer in Eqs. (30)–(32), $\hat{\mathbf{X}}_k$ will be obtained as

$$\begin{aligned} \hat{\mathbf{X}}_k &= \hat{\mathbf{X}}_k^- + \mathbb{K}_k\cdot(\mathbf{Y}_k - \mathrm{dual}(C_k\hat{\mathbf{X}}_k^-)) \\ \mathbf{Var}(\hat{\mathbf{X}}_k) &= \begin{pmatrix} \mathbf{Var}_{\mathbf{xx}}(\hat{\mathbf{X}}_k) \\ \mathbf{Var}_{\mathbf{x}\cdot\mathrm{dual}\mathbf{x}}(\hat{\mathbf{X}}_k) \end{pmatrix} \end{aligned} \tag{34}$$

where

$$\begin{aligned} \mathbb{K}_k &= \mathbf{Var}_{\mathbf{x}\cdot\mathrm{dual}\mathbf{x}}(\hat{\mathbf{X}}_k^-)\cdot C_k^{\mathrm{T}} \\ &\quad \cdot \mathrm{dual}(C_k\cdot \mathbf{Var}_{\mathbf{xx}}(\hat{\mathbf{X}}_k^-)\cdot C_k^{\mathrm{T}} + \mathbf{Var}_{\mathbf{vv}}(\mathbf{v}_k))^{-1} \\ \mathbf{Var}_{\mathbf{xx}}(\hat{\mathbf{X}}_k) &= \mathbf{Var}_{\mathbf{xx}}(\hat{\mathbf{X}}_k^-) - \mathrm{dual}(\mathbb{K}_k\cdot C_k\cdot \mathbf{Var}_{\mathbf{x}\cdot\mathrm{dual}\mathbf{x}}(\hat{\mathbf{X}}_k^-)) \\ &\quad - \mathbf{Var}_{\mathbf{x}\cdot\mathrm{dual}\mathbf{x}}(\hat{\mathbf{X}}_k^-)\cdot C_k^{\mathrm{T}}\cdot \mathrm{dual}\mathbb{K}_k^{\mathrm{T}} \\ &\quad + \mathrm{dual}\mathbb{K}_k\cdot C_k\cdot \mathrm{dual}\mathbf{Var}_{\mathbf{xx}}(\hat{\mathbf{X}}_k^-)\cdot C_k^{\mathrm{T}}\cdot \mathrm{dual}\mathbb{K}_k^{\mathrm{T}} \\ \mathbf{Var}_{\mathbf{x}\cdot\mathrm{dual}\mathbf{x}}(\hat{\mathbf{X}}_k) &= \mathbf{Var}_{\mathbf{x}\cdot\mathrm{dual}\mathbf{x}}(\hat{\mathbf{X}}_k^-) - \mathbf{Var}_{\mathbf{xx}}(\hat{\mathbf{X}}_k^-)\cdot C_k^{\mathrm{T}}\cdot \mathbb{K}_k^{\mathrm{T}} \\ &\quad - \mathrm{dual}(\mathbb{K}_k\cdot C_k\cdot \mathbf{Var}_{\mathbf{xx}}(\hat{\mathbf{X}}_k^-)) \\ &\quad + \mathrm{dual}\mathbb{K}_k\cdot C_k\cdot \mathrm{dual}\mathbf{Var}_{\mathbf{x}\cdot\mathrm{dual}\mathbf{x}}(\hat{\mathbf{X}}_k^-)\cdot C_k^{\mathrm{T}}\cdot \mathbb{K}_k^{\mathrm{T}} \end{aligned}$$

The correction term in Eq. (34) depends on the residual $\mathbf{Y}_k - \mathrm{dual}(C_k\hat{\mathbf{X}}_k^-)$. The residual coefficient $\mathbb{K}_k$ is equivalent to the Kalman gain K_k in Eq. (4). It has an expression in terms of the priori error covariance $\mathbf{Var}(\hat{\mathbf{X}}_k^-)$.

4 Numerical Examples

In this section, two examples are given to illustrate the proposed mechanism. The first one is a simple numerical example, which is to estimate a random constant value. The second one is an application to the estimation of dimensional variations in a multistage mechanical assembly process.

4.1 Example 1: Estimating a Random Constant Value. A simple example is presented here to illustrate the operation and capability of the proposed mechanism. The example is adopted from Ref. [52], which was used to illustrate the classical Kalman filter. Suppose a scalar random constant value of voltage is to be estimated, subject to measurement noise and a small external adjustment. The state model in Eqs. (1) and (2) becomes

$$X_k = X_{k-1} + U_k + w \qquad Y_k = X_k + v \tag{35}$$

where the system state does not change by itself from step to step with $A_k = 1$, a very small control input is applied with $B = 1$ and $U_k = -0.001$, the measurement is applied directly to the state with $C_k = 1$, $X_0 \sim N(0,\mathrm{Var}(X_0))$, $w \sim N(0,\mathrm{Var}(w))$, and $v \sim N(0, \mathrm{Var}(v))$.

When the model is extended to consider interval uncertainty, Eq. (35) becomes

$$\mathbf{X}_k = \mathbf{X}_{k-1} + \mathbf{U}_k + w \qquad \mathbf{Y}_k = \mathbf{X}_k + \mathbf{v} \tag{36}$$

where $\mathbf{X}_0 \sim \mathbb{N}(0, \mathbf{Var}(\mathbf{X}_0))$, $\mathbf{w} \sim \mathbb{N}(0, \mathbf{Var}(\mathbf{w}))$, and $\mathbf{v} \sim \mathbb{N}(0, \mathbf{Var}(\mathbf{v}))$. The control input has an improper interval value $\mathbf{U}_k = [-0.0008, -0.0012]$, which implies that the input is controllable and adjustable within the bound. Again, more details about generalized interval and its logic interpretations can be found in Refs. [29,43,53]. The data used in Example 1 are listed in Table 1, in which the values in the second column are the original ones from Ref. [52].

Based on Eqs. (3), (4), (18), and (34), we can calculate the prediction and update steps for the classical Kalman filter and the proposed mechanism, respectively. In the simulation, a randomly generated value $X = -0.37727$ is first chosen to represent the true value. One hundred measurements of Y_k values are then sampled based on a normal distribution with the mean $X = -0.37727$ and the standard deviation of 0.1. In Fig. 2(*a*), the true value is shown as the dashed line. The sampled 100 measurements are plotted as the crosses sequentially. The dash–dot curve denotes the sequence of estimated values by the classical Kalman filter. The estimated value becomes very close to the true one after 50 iterations, as shown in Fig. 2(*a*), assuming that the distributions are precisely known.

For the proposed mechanism, the measurements' $\mathbf{Y}_k$ values are derived from previous Y_k values with ± 10 variation, with the consideration of systematic error. The imprecise measurements are represented by the short solid lines in Fig. 2(*b*). The lower and upper bounds of the estimated mean values by the proposed mechanism are shown by the solid curves. It is seen that the estimation from the classical Kalman filter is always included by the one from the proposed mechanism after the initial transient stage. In Fig. 2(*c*), the variance produced by the classical Kalman filter is also included by the proposed method. The lower bound is very close to the one from the classical Kalman filter when the number of iterations increases, as shown in the enlarged plot. The lower and upper bounds are approaching each other, and they converge to one from the classical Kalman filter quickly and almost become identical after 90 iterations.

It is also observed that the estimation of the mean from the classical Kalman filter fluctuates above or below the true value, particularly at the beginning. Thus, the estimation either over- or underestimate the true value, depending on the number of iterations for observation before the estimation is taken. In contrast, the interval estimation in the proposed mechanism always encloses the true value at each iteration. The estimated interval mean values gradually shrink towards the true value. The convergence speed of the proposed method is slower than the classical one, which avoids the fluctuation.

Table 1 Data used in Example 1

Data	Classical Kalman filter	Proposed mechanism
$\mathbf{X}_0$	0	0
$\mathbf{Var}(\mathbf{X}_0)$	1	$([0.9, 1.1], 1)^{\mathrm{T}}$
$\mathbf{Var}(\mathbf{w})$	$1e^{-5}$	$([1e^{-6}, 1e^{-4}], 1e^{-5})^{\mathrm{T}}$
$\mathbf{Var}(\mathbf{v})$	0.01	$([0.009, 0.011], 0.01)^{\mathrm{T}}$

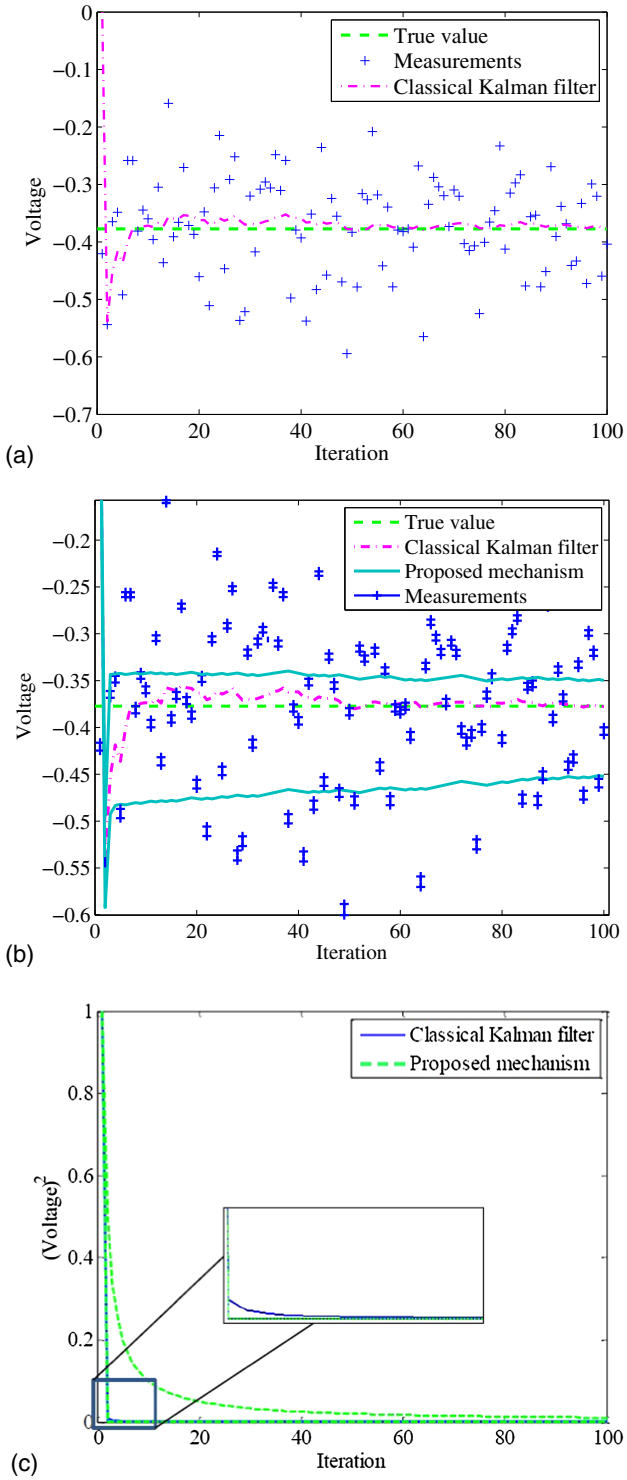

Fig. 2 Comparison between the results from the classical Kalman filter and the proposed mechanism in Example 1: (*a*) estimated voltage values of the classical Kalman filter, (*b*) mean value comparison, and (*c*) variance comparison

The proposed mechanism extends the classical Kalman filter with the consideration of both random error and systematic error associated with the measurements with improved robustness, whereas the classical Kalman filter only models random error. With the consideration of bias in experimental measurements, interval probability distributions eliminate the assumption of Gaussian distributions. In other words, the sensitivity information of what if the noise deviates from a Gaussian distribution can be obtained efficiently with the new mechanism, where the interval bounds estimate the effect of deviation. If sensitivity analysis is required in the classical Kalman filter, samples need to be drawn for each of the assumed distributions for X_0, w, and v. In contrast, in the proposed mechanism, one run of calculations can provide similar information. In addition, the variance may be underestimated in the classical Kalman filter because of the deviation away from a Gaussian distribution. In the new mechanism, interval variance provides an estimation of a range of values. Therefore, the proposed method can provide more robust estimations of the mean and variance than the classical Kalman filter in a more efficient manner.

Interval arithmetic tends to overestimate when variables appear multiple times in functions. In the formulation of the update step in the proposed method, there are multiple occurrences of variables. Thus, the results from the proposed method are likely to be an overestimation when all of the intervals are proper. Nevertheless, a moderate level of overestimation is acceptable if robustness is preferred.

4.2 Example 2: Modeling Dimensional Variation in Mechanical Assemblies. The dimensional variation propagation in a multistage assembly process can be described as a linear discrete-time dynamic system, where "time" is the stage as an independent variable. The uncertainty propagating along the assembly process causes dimensional variations of parts from their specified values. Random errors arise from environmental disturbance, inhomogeneity of materials, or nonuniformity of components. Systematic errors come from model simplification, measurement bias, and nonobservability. For example, a measured dimension x can never be quantified with absolute certainty; only an interval value $\mathbf{x} = [\underline{x}, \bar{x}]$ is obtained. It could be caused by the measuring limitation of equipment, round off during data digitization, personal bias, and so on. The measured quantity is actually a random interval instead of a precise one, given the various sources of systematic errors. In this case, the distribution of measurement $\mathbf{x}$ values has the imprecise mean and variance, represented as interval numbers. If the interval nature of $\mathbf{x}_i$ is neglected, the dimensional fluctuation can be underestimated. Therefore, it is useful to apply interval-valued probability distributions in analyzing dimensional variations with the effects of systematic error and random error differentiated.

In this example, the multistage assembly process with the consideration of measurement error is used to demonstrate the proposed method. The example is adopted from Refs. [54–56], which was formulated by considering traditional statistical distributions. The multistage assembly model is obtained from a side aperture assembly line in the automotive industry. As in Fig. 3, four stages $(S_1, \ldots, S_4)$ are shown. All parts are assumed to be rigid and variations only occur in the $x-z$ plane. A total of eight sensing and measurement points $(m_1, \ldots, m_8)$ are available. Each measurement has the coordinate values of x and z. It is also assumed that all of the parts are manufactured at the same location, and the dimensional deviations follow a distribution with zero means and interval-valued variances.

The state-space model for this four-stage assembly process is formulated as

$$\mathbf{X}_i = A_i \mathbf{X}_{i-1} + B_i \mathbf{U}_i + \mathbf{w}_i \quad i = 1, 2, 3 \tag{37}$$

$$\mathbf{X}_4 = A_4 \mathbf{X}_3 + \mathbf{w}_4 \tag{38}$$

$$\mathbf{Y} = C\mathbf{X}_4 + \mathbf{v} \tag{39}$$

where $\mathbf{X}_0 \in \mathbb{KR}^{12\times 1}$ is a 12-dimensional vector that represents the initial dimensional variations associated with the incoming parts, and $\mathbf{Y} \in \mathbb{KR}^{16\times 1}$ corresponds to the x- and z-coordinate measurements at the eight measurement points at the fourth stage. The values of matrices A_i, B_i, and C are given in Ref. [56].

For the proposed method, input $\mathbf{X}_0$ is set to be distributed as $\mathbb{N}(0, \mathbf{Var}(\mathbf{X}_0))$, where $\mathbf{Var}(\mathbf{X}_0) = [\mathbf{Var}_{\mathbf{xx}}(\mathbf{X}_0) \quad \mathbf{Var}_{\mathbf{xdualx}}(\mathbf{X}_0)]^{\mathrm{T}}$.

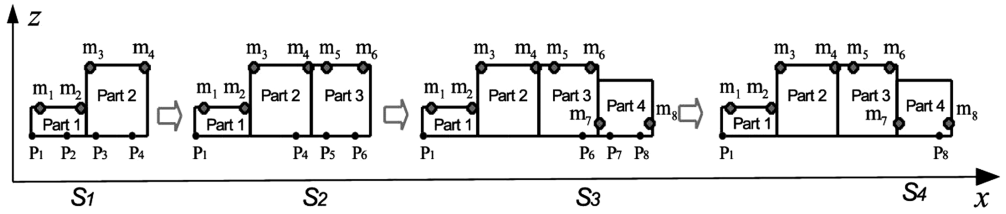

Fig. 3 Assembly sequence, locating, and measurement points in Example 2

$\mathbf{Var_{xx}}(\mathbf{X}_0)$ is assumed to vary ± 10 of the nominal value $0.04 \cdot I$ with the consideration of systematic error, where I stands for the identity matrix with appropriate dimensions. $\mathbf{Var_{xdualx}}(\mathbf{X}_0)$ with ± 5 variation is also assumed because of $\mathbf{Var_{xdualx}}(\mathbf{X}_0) \subseteq \mathbf{Var_{xx}}(\mathbf{X}_0)$ and the symmetry of measurement error. The same assumptions are also applied to $\mathbf{Var}(\mathbf{w})$ and $\mathbf{Var}(\mathbf{v})$, where the nominal values are $0.0001 \cdot I$ and $0.0009 \cdot I$, respectively. $\mathbf{U}$ is controlled within the range of $[\![-10, 10]\!]$ by the tooling locators. The unit for all values is millimeters.

To verify the robustness of interval estimations, a Monte Carlo sampling procedure is taken, where real-valued variances are sampled to run the classical Kalman filter. The real-valued variances $\mathrm{Var}(X_0)$ for initial inputs X_0 are sampled uniformly within the corresponding interval ranges. For each of the 12 locators, 100 samples are taken as initial variances for the classical Kalman filter formulations in Eqs. (3) and (4). That is, 100 possible distributions are tested for the dimensional variations for each of the 12 locators. The sampled real-valued variances are then computed in the classical Kalman filter formulation to predict the dimensional variation. Because the assembly line has three assembly stations and one measurement station, there are four iterated prediction steps and one update step in the calculation. Because $\mathbf{Var_{xx}}(\mathbf{X}_0)$ is comparable with traditional $\mathrm{Var}(X_0)$, $\mathrm{Var}(X_0)$ is sampled within $\mathbf{Var_{xx}}(\mathbf{X}_0)$, whereas $\mathbf{Var_{xdualx}}(\mathbf{X}_0))$ is not used in the sampling. In addition, $U \sim N(0, 0.0017)$, $w \sim N(0, 0.0001)$, and $v \sim N(0, 0.0009)$ are also assumed in the verification. One hundred samples are drawn for each of w and v at a time. The noises are assumed to be the same for all four stages.

The variances estimated from the proposed mechanism and the ones from the classical Kalman filter are compared in Fig. 4, in which the cross markers (+) denote the ones computed by the classical Kalman filter during the Monte Carlo sampling, and circles denote the bounds of $\mathbf{Var_{xx}}(\mathbf{X})$ from the proposed mechanism. As shown in the figure, the dimensional variations of locators with real-valued $\mathrm{Var}(X)$ are sampled by the Monte Carlo method and propagated from station 1 to station 4 by the prediction step. The variances along with each stations from the classical Kalman filter are always enclosed by $\mathbf{Var_{xx}}(\mathbf{X})$, which are computed by the

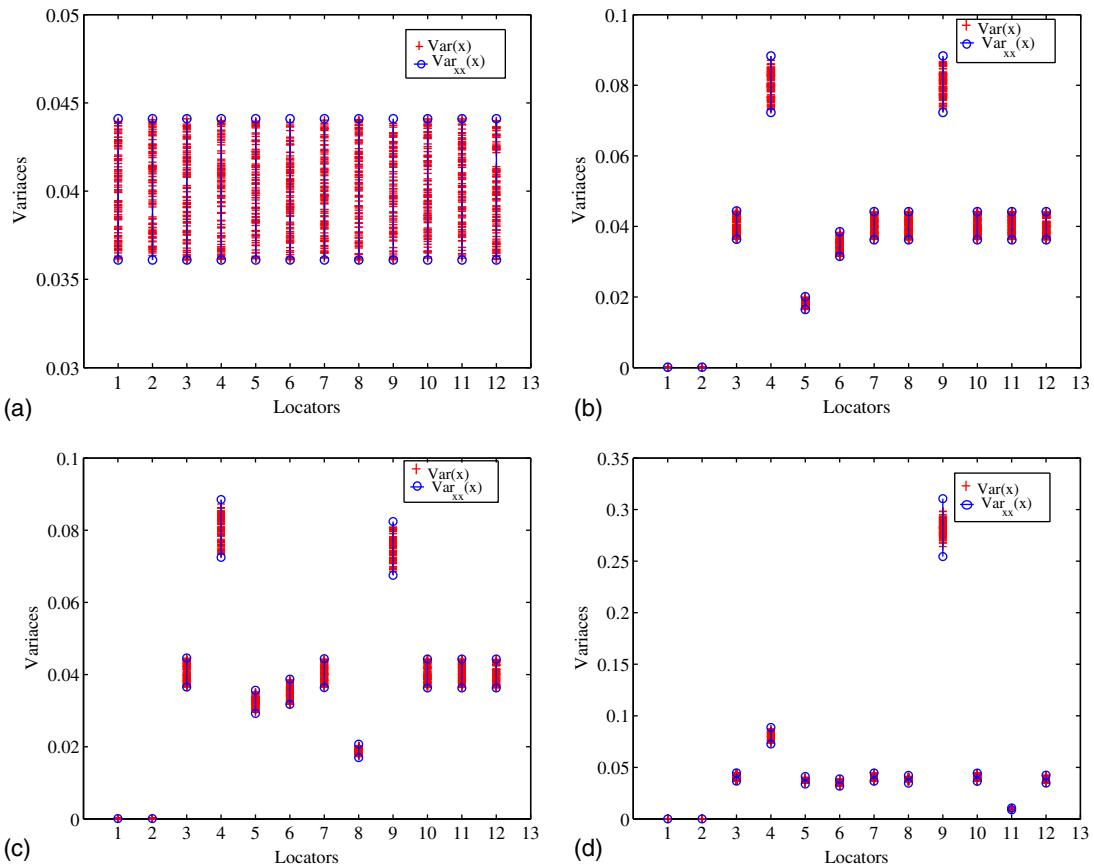

Fig. 4 Results comparison between classical and the proposed Kalman filter: (*a*) variances estimated at stage 1, (*b*) variances estimated at stage 2, (*c*) variances estimated at stage 3, and (*d*) variances estimated at stage 4

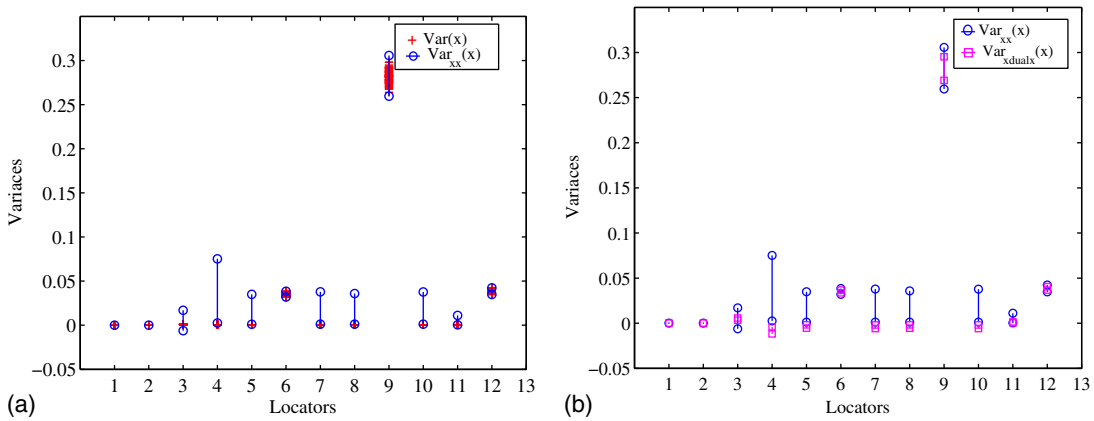

Fig. 5 Variances updated at stage 4: (*a*) variances comparison between the two methods and (*b*) variances computed by the proposed method

prediction step of the proposed mechanism. The samples of $\mathrm{Var}(X)$ are given as uniformly distributed within interval ranges, as shown in Fig. 4(*a*). The uniformity is preserved until station 3, as shown in Figs. 4(*b*) and 4(*c*). The cross markers are uniformly distributed within the entire interval range. At station 4, the uniformity is changed at locator 9, where $\mathbf{Var_{xx}}(\mathbf{X})$ is overestimated compared to the Monte Carlo sampling, as shown in Fig. 4(*d*). At other locators, reasonable completeness and soundness of the interval estimations are seen. If considering all possible combinations of sampled noises and variances of 12 locators, the number of simulation runs for the classical Kalman filter will be 100^{12}, which is not computable. The efficiency of the proposed mechanism is obvious. One run of the interval-based method can provide the range estimation that requires many runs of the classical method, if sensitivity analysis is applied to test the robustness.

The results of the update at stage 4 are shown in Fig. 5. A comparison between $\mathrm{Var}(X)$ and $\mathbf{Var_{xx}}(\mathbf{X})$ is shown in Fig. 5(*a*), in which $\mathrm{Var}(X)$ are always enclosed by $\mathbf{Var_{xx}}(\mathbf{X})$. However, the interval range $\mathbf{Var_{xx}}(\mathbf{X})$ is much larger than the area covered by the values of $\mathrm{Var}(X)$. There could be two reasons for this. One is the underestimation from the classical Kalman filter, which do not consider all possible combinations of sampled noise and variances. The other is the overestimation from the update step of the proposed method because of the multiple occurrences of variables. Note that $\mathbb{K}_k$ appears four times, $\mathbf{Var_{xdualx}}(\hat{\mathbf{X}}_k^-)$ appears twice, and $\mathbf{Var_{xx}}(\hat{\mathbf{X}}_k^-)$ appears twice in $\mathbf{Var_{xx}}(\hat{\mathbf{X}}_k)$ in Eq. (34).

The values of $\mathbf{Var_{xx}}(\mathbf{X})$ and $\mathbf{Var_{xdualx}}(\mathbf{X})$ computed by the proposed method are shown in Fig. 5(*b*). Some values of $\mathbf{Var_{xdualx}}(\mathbf{X})$ are negative. Based on Definition 2 and Theorem 2, $\mathbf{Var_{xdualx}}(\mathbf{X})$ measures the covariance between the lower and upper bounds. Negative values reveal that the upper and lower bounds of interval-valued dimensions vary in the opposite trend. A negative value of $\mathbf{Var_{xx}}(\mathbf{X})$ also occurs at locator 3. This is likely to be caused by the overestimation in the inverse interval matrix when calculating $\mathbb{K}$ in Eq. (33). The overestimation of interval matrix $\mathbb{K}$ can be passed over to $\mathbf{Var_{xx}}(\mathbf{X})$ as in Eq. (34). Currently, the inverse of matrix $\mathbf{A} = [\underline{A}, \bar{A}]$ is simply calculated as $\mathbf{A}^{-1} = [\underline{A}^{-1}, \bar{A}^{-1}]$. Other ways to compute the inverse of interval matrices to reduce overestimation are needed in future studies.

Given the systematic error associated with measurement and the initial variance estimation, the proposed method is an efficient alternative to sensitivity analysis. For each of the assumed initial distributions of state vector, noise, and observations, Monte Carlo sampling is the typical way to assess the effect of error in the classical Kalman filter. High computational costs come from running the model with combinations of different input values to realize all possible variations. In contrast, the interval estimation provides similar information with only one run of calculations.

5 Concluding Remarks

In this paper, an extended Kalman filter mechanism which differentiates systematic error from random error is proposed. It is based on generalized interval probability that integrates generalized interval and probability. In this approach, state and observation variables are random intervals. Noise is modeled as interval Gaussian distributions. The variance of random intervals is defined as both the scattering of the interval bounds and the covariance between the lower and upper bounds. The new mechanism is an alternative to sensitivity analysis to quantify the effect of systematic error in the inputs of the Kalman filter. The two examples show that the estimations of interval means and variances in the proposed method can provide more robust prediction than the classical Kalman filter with the consideration of measurement errors. The prediction of interval values can provide more information than traditional precise ones, such as the best-case and worst-case scenarios, the effect of systematic error, and the convergence speed of estimation when imprecise data are applied.

In the future, the range overestimation associated with interval analysis in the proposed method must be addressed. Both the efficiency of computation and the value of the information provided by interval ranges need to be considered. The overestimation of interval analysis is typically caused by multiple occurrences of variables in functions. The interval extension of a real-valued function in which all variables are in continuous domains is usually defined by replacing real operands and operators by the corresponding interval ones, in which multiple occurrences are inherited. On the other side, overestimation is an outer estimation which intends to provide a complete range in which all possible values are included. The inner estimation that provides a sound range also needs to be studied such that more information is available. The goal is to provide a more accurate prediction of hidden states while maintaining good robustness and computational efficiency.

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