Imprecise probabilities based on generalized intervals for system reliability assessment

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Abstract: Different representations of imprecise probabilities have been proposed, where interval-valued probabilities are used such that uncertainty is distinguished from variability. In this paper, we present a new form of imprecise probabilities for reliability assessment based on generalized intervals. Generalized intervals have group properties under the Kaucher arithmetic, which provides a concise representation and calculus structure as an extension of precise probabilities.

With the separation between proper and improper interval probabilities, *focal* and *non-focal* events are differentiated based on the associated modalities and logical semantics. Focal events have the semantics of critical, uncontrollable, and specified in probabilistic analysis, whereas the corresponding non-focal events are complementary, controllable, and derived.

A logic coherence constraint is proposed in the new form. Because of the algebraic properties of generalized intervals, conditional interval probability can be directly defined based on marginal interval probabilities. A Bayes' rule with generalized intervals allows us to interpret the logic relationship between interval prior and posterior probabilities. The imprecise Dirichlet model is also extended with the logic coherence constraint.

Keywords: interval arithmetic; generalized interval; imprecise probability; imprecise Dirichlet model; conditioning; updating.

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1 Introduction

Imprecise probability differentiates uncertainty from variability both qualitatively and quantitatively, which is the alternative to the traditional sensitivity analysis in probabilistic reasoning to model indeterminacy and imprecision. Many representations of imprecise probabilities have been developed in the past four decades. The Dempster-Shafer evidence theory (Dempster (1967); Shafer (1976)) characterizes uncertainties as discrete probability masses associated with a power set of values. Belief-Plausibility pairs are used to measure likelihood. The behavioral imprecise probability theory (Walley (1991)) models behavioral uncertainties with the lower prevision (supremum acceptable buying price) and the upper prevision (infimum acceptable selling price). A random set (Molchanov (2005)) is a multi-valued mapping from the probability space to the value space. The possibility theory (Zadeh (1978); Dubois and Prade (1988)) provides an alternative to represent uncertainties with Necessity-Possibility pairs. Probability bound analysis (Ferson et al. (2002)) captures uncertain information with p-boxes which are pairs of lower and upper probability distributions. F-probability (Weichselberger (2000)) incorporates intervals into the probability value which maintains Kolmogorov properties. Fuzzy probability (Möller and Beer (2004)) considers probability distributions with fuzzy parameters. A cloud (Neumaier (2004)) is a fuzzy interval with an interval-valued membership, which is a combination of fuzzy sets, intervals, and probability distributions.

These different representations model the indeterminacy due to incomplete information very well with different forms. There are still challenges in practical issues such as assessment and computation to derive inferences and conclusions (Walley (1996a)). For instance, computing the lower and upper envelopes from the extremes of interval-valued probabilities in belief updating and inference is complex. Usually this combinatorial problem is formulated and solved by linear programming, which requires a polynomial time of computation. A simpler algebraic structure of imprecise probability will be helpful in extending its applications in engineering and science domains, where intuitive calculus often shows advantages in ease of use and reducing chances of human errors. We recently proposed a new form of imprecise probabilities based on generalized intervals (Wang (2008)). Unlike traditional set-based intervals, such as the interval $[0.1, 0.2]$ which represents a set of real values between 0.1 and 0.2, generalized intervals also allow the existence of the interval $[0.2, 0.1]$. The logic quantifiers ($\forall$ and $\exists$) can be integrated to provide the interpretation of intervals. Another advantage of generalized intervals is that they have group properties under arithmetic operations $(+, -, \times, \div)$. With this extension, the algebraic properties of interval-valued probabilities are improved, and the calculus is simplified.

In this paper, we demonstrate that the new interval probability structure can be applied in reliability assessment. In reliability analysis, the motivations of using imprecise probabilities include lack of statistical data to generate precise distributions, subjective judgements from experts causing inconsistencies or conflicts, lack of knowledge about physical systems such as ageing effects and dependency relationships among components, and imprecision of measurements such as censored data, which cannot be modeled efficiently in traditional analysis with precise probabilities (Coolen (2004); Utkin and Coolen (2007)). Imprecise probability has attracted reliability researchers' attentions in the past decade. Coolen and Newby (1994) showed that the application of imprecise probabilities can make the elicitation of prior information from experts simpler by considering a range of possible probabilities. Coolen (1997) introduced the Bayesian analysis with imprecise Dirichlet model (Walley (1996b); Bernard (2005)) into the prediction of failure rates. Non-parametric predictive inference approaches (Coolen (1998); Coolen and Yan (2004); Coolen-Schrijner and Coolen (2007); Coolen and Augustin (2007)) to predict future failures based on past observations were also developed, which can be applied to right-censored data and support preventive replacement decisions. Utkin and Gurov (1999, 2002) studied the reliability of monotone multistate systems based on a generic setting of non-parametric life distribution classes and the natural extension constraints in Kuznetsov's dual form (Utkin and Kozine (2001)). Interval reliabilities of parallel and series systems were derived (Utkin (2004)).

Different forms of imprecise probabilities have been used in reliability analysis. For example, Kozine and Filimonov (2000) compared the applications of the Dempster-Shafer structure and the coherent imprecise probability theory in reliability assessment. Tonon et al. (1999, 2000) applied random sets and evidence theory for structure reliability analysis. Nikolaidis et al. (2004) and Soundappan et al. (2004) used the evidence and possibility theories in robust design under uncertainties against failures. The evidence and possibility theories were also employed to formulate and solve reliability based design optimization problems (Mourelatos and Zhou (2006); Du et al. (2006); Kokkolaras et al. (2006); Zhou and Mourelatos (2008)). Whitcomb (2005) applied the generalized Bayes' rule to mean time to failure estimations in conjunction with linear programming under coherence constraints. Coherent imprecise probabilities have also been applied in studies of cold standby systems (Utkin (2003b)) and bridge system structures (Song et al. (2006)). Aughenbaugh and Herrmann (2008) combined the imprecise Dirichlet model with the variance-based sensitivity indices (Hall (2006)) and applied to reliability test planning.

One of the core issues in imprecise probability is to characterize incomplete knowledge of distributions with lower and upper probability pairs so that we can improve the robustness of assessment. We are interested in exploring the potential of our new imprecise probability structure applied to reliability analysis. In the remainder of this paper, Section 2 summarizes the algebraic and logic properties of generalized intervals. Section 3 presents the new imprecise probability structure with the generalized interval form. Section 4 shows the extended imprecise Dirichlet model and demonstrates its application in reliability assessment. Finally Section 5 is the Conclusion.

2 Generalized Intervals

Our new imprecise probability structure is based on generalized or modal intervals. Modal interval analysis (MIA) (Gardeñes et al. (2001); Markov (2001); Shary (2002); Popova (2001); Armengol et al. (2001)) is an algebraic and semantic extension of the classical interval analysis (IA) (Moore (1966)). In IA, an interval $[a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}$ is a set of real numbers defined by its lower and upper bounds. Therefore, the interval $[a, b]$ becomes invalid or empty when $a > b$. In MIA, a generalized interval is no longer restricted to the ordered bound condition of $a \leq b$. Therefore, $[2, 1]$ is also a valid interval.

A generalized interval $\mathbf{x} := [\underline{x}, \overline{x}]$ $(\underline{x}, \overline{x} \in \mathbb{R})$ is defined by a pair of real numbers $\underline{x}$ and $\overline{x}$, instead of the traditional set-based definition. The generalized interval

Table 1 The Kaucher arithmetic

$$\mathbf{x}+\mathbf{y} := \left[\underline{x}+\underline{y}, \overline{x}+\overline{y}\right]$$

$$\mathbf{x}-\mathbf{y} := \left[\underline{x}-\overline{y}, \overline{x}-\underline{y}\right]$$

$$\mathbf{x}\times\mathbf{y} := \begin{cases}
\left[\underline{x}\underline{y}, \overline{x}\overline{y}\right] & \left(\underline{x}\geq 0, \overline{x}\geq 0, \underline{y}\geq 0, \overline{y}\geq 0\right)\\
\left[\underline{x}\underline{y}, \underline{x}\overline{y}\right] & \left(\underline{x}\geq 0, \overline{x}\geq 0, \underline{y}\geq 0, \overline{y}< 0\right)\\
\left[\overline{x}\underline{y}, \overline{x}\overline{y}\right] & \left(\underline{x}\geq 0, \overline{x}\geq 0, \underline{y}< 0, \overline{y}\geq 0\right)\\
\left[\overline{x}\underline{y}, \underline{x}\overline{y}\right] & \left(\underline{x}\geq 0, \overline{x}\geq 0, \underline{y}< 0, \overline{y}< 0\right)\\
\left[\underline{x}\underline{y}, \overline{x}\underline{y}\right] & \left(\underline{x}\geq 0, \overline{x}< 0, \underline{y}\geq 0, \overline{y}\geq 0\right)\\
\left[\max\left(\underline{x}\underline{y}, \overline{x}\overline{y}\right), \min\left(\underline{x}\overline{y}, \overline{x}\underline{y}\right)\right] & \left(\underline{x}\geq 0, \overline{x}< 0, \underline{y}\geq 0, \overline{y}< 0\right)\\
[0,0] & \left(\underline{x}\geq 0, \overline{x}< 0, \underline{y}< 0, \overline{y}\geq 0\right)\\
[\overline{x}\overline{y}, \underline{x}\overline{y}] & \left(\underline{x}\geq 0, \overline{x}< 0, \underline{y}< 0, \overline{y}< 0\right)\\
[\underline{x}\overline{y}, \overline{x}\overline{y}] & \left(\underline{x}< 0, \overline{x}\geq 0, \underline{y}\geq 0, \overline{y}\geq 0\right)\\
[0,0] & \left(\underline{x}< 0, \overline{x}\geq 0, \underline{y}\geq 0, \overline{y}< 0\right)\\
\left[\min\left(\underline{x}\overline{y}, \overline{x}\underline{y}\right), \max\left(\underline{x}\underline{y}, \overline{x}\overline{y}\right)\right] & \left(\underline{x}< 0, \overline{x}\geq 0, \underline{y}< 0, \overline{y}\geq 0\right)\\
\left[\overline{x}\underline{y}, \underline{x}\underline{y}\right] & \left(\underline{x}< 0, \overline{x}\geq 0, \underline{y}< 0, \overline{y}< 0\right)\\
\left[\underline{x}\overline{y}, \overline{x}\underline{y}\right] & \left(\underline{x}< 0, \overline{x}< 0, \underline{y}\geq 0, \overline{y}\geq 0\right)\\
\left[\overline{x}\overline{y}, \overline{x}\underline{y}\right] & \left(\underline{x}< 0, \overline{x}< 0, \underline{y}\geq 0, \overline{y}< 0\right)\\
\left[\underline{x}\overline{y}, \underline{x}\underline{y}\right] & \left(\underline{x}< 0, \overline{x}< 0, \underline{y}< 0, \overline{y}\geq 0\right)\\
\left[\overline{x}\overline{y}, \underline{x}\underline{y}\right] & \left(\underline{x}< 0, \overline{x}< 0, \underline{y}< 0, \overline{y}< 0\right)
\end{cases}$$

$$\mathbf{x}/\mathbf{y} := \begin{cases}
\left[\underline{x}/\overline{y}, \overline{x}/\underline{y}\right] & \left(\underline{x}\geq 0, \overline{x}\geq 0, \underline{y}> 0, \overline{y}> 0\right)\\
\left[\overline{x}/\overline{y}, \underline{x}/\underline{y}\right] & \left(\underline{x}\geq 0, \overline{x}\geq 0, \underline{y}< 0, \overline{y}< 0\right)\\
[\underline{x}/\overline{y}, \overline{x}/\overline{y}] & \left(\underline{x}\geq 0, \overline{x}< 0, \underline{y}> 0, \overline{y}> 0\right)\\
\left[\overline{x}/\underline{y}, \underline{x}/\underline{y}\right] & \left(\underline{x}\geq 0, \overline{x}< 0, \underline{y}< 0, \overline{y}< 0\right)\\
\left[\underline{x}/\underline{y}, \overline{x}/\underline{y}\right] & \left(\underline{x}< 0, \overline{x}\geq 0, \underline{y}> 0, \overline{y}> 0\right)\\
[\overline{x}/\overline{y}, \underline{x}/\overline{y}] & \left(\underline{x}< 0, \overline{x}\geq 0, \underline{y}< 0, \overline{y}< 0\right)\\
\left[\underline{x}/\underline{y}, \overline{x}/\overline{y}\right] & \left(\underline{x}< 0, \overline{x}< 0, \underline{y}> 0, \overline{y}> 0\right)\\
\left[\overline{x}/\underline{y}, \underline{x}/\overline{y}\right] & \left(\underline{x}< 0, \overline{x}< 0, \underline{y}< 0, \overline{y}< 0\right)
\end{cases}$$

is related to the traditional set-based interval by an operator $'$ as in $[\underline{x}, \overline{x}]' := \{x \in \mathbb{R} | \min(\underline{x}, \overline{x}) \leq x \leq \max(\underline{x}, \overline{x})\}$. $\mathbf{x}$ is called *proper* when $\underline{x} \leq \overline{x}$ and called *improper* when $\underline{x} \geq \overline{x}$. When $\underline{x} = \overline{x}$, $\mathbf{x}$ is a *pointwise* interval. Pointwise intervals are both proper and improper. We also denote a pointwise interval $[x, x]$ simply as x. The set of generalized intervals is denoted by $\mathbb{KR} = \{[\underline{x}, \overline{x}] \mid \underline{x}, \overline{x} \in \mathbb{R}\}$. The set of proper intervals is denoted by $\mathbb{IR} = \{[\underline{x}, \overline{x}] \mid \underline{x} \leq \overline{x}\,(\underline{x}, \overline{x} \in \mathbb{R})\}$, and the set of improper interval by $\overline{\mathbb{IR}} = \{[\underline{x}, \overline{x}] \mid \underline{x} \geq \overline{x}\,(\underline{x}, \overline{x} \in \mathbb{R})\}$. Operations between two generalized intervals $\mathbf{x} = [\underline{x}, \overline{x}]$ and $\mathbf{y} = [\underline{y}, \overline{y}]$ are defined in the Kaucher arithmetic (Kaucher (1980)) as in Table 1.

Given a generalized interval $\mathbf{x} = [\underline{x}, \overline{x}] \in \mathbb{KR}$, two operators *pro* and *imp* return proper and improper values, defined as $\text{pro}\mathbf{x} := [\min(\underline{x}, \overline{x}), \max(\underline{x}, \overline{x})]$ and $\text{imp}\mathbf{x} := [\max(\underline{x}, \overline{x}), \min(\underline{x}, \overline{x})]$ respectively. The relationship between proper and improper intervals is established with the operator *dual* as $\text{dual}\mathbf{x} := [\overline{x}, \underline{x}]$.

For example, $\mathbf{a} = [0, 1]$ and $\mathbf{b} = [1, 0]$ are both valid intervals. While $\mathbf{a}$ is a proper interval, $\mathbf{b}$ is an improper one. The relation between $\mathbf{a}$ and $\mathbf{b}$ can be established by $\mathbf{a} = \text{dual}\mathbf{b}$ and $\mathbf{b} = \text{dual}\mathbf{a}$. The *inclusion* relationship $\subseteq$ between generalized intervals $\mathbf{x} = [\underline{x}, \overline{x}]$ and $\mathbf{y} = [\underline{y}, \overline{y}]$ is defined as

(1) $$[\underline{x}, \overline{x}] \subseteq [\underline{y}, \overline{y}] \iff \underline{x} \geq \underline{y} \wedge \overline{x} \leq \overline{y}$$

The *less-than-or-equal-to* relationship $\leq$ is defined as

(2) $$[\underline{x}, \overline{x}] \leq [\underline{y}, \overline{y}] \iff \underline{x} \leq \underline{y} \wedge \overline{x} \leq \overline{y}$$

Unlike IA which identifies an interval by a set of real numbers only, MIA identifies an interval by a set of predicates which is fulfilled by real numbers. Given a set of closed intervals of real numbers, and the set of logical existential ($\exists$) and universal ($\forall$) quantifiers, each generalized interval has an associated quantifier. The semantics of a generalized interval $\mathbf{x} \in \mathbb{KR}$ is denoted by $(\mathrm{Q}_{\mathbf{x}} x \in \mathbf{x}')$ where $\mathrm{Q}_{\mathbf{x}} \in \{\exists, \forall\}$. $\mathbf{x}$ is called *existential* if $\mathrm{Q}_{\mathbf{x}} = \exists$. Otherwise, it is called *universal*. If a real relation $z = f(x_1, \ldots, x_n)$ is extended to the interval relation $\mathbf{z} = \mathbf{f}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$, the interval relation $\mathbf{z}$ is interpretable if there is a semantic relation

(3) $$(\mathrm{Q}_{\mathbf{x}_1} x_1 \in \mathbf{x}_1') \cdots (\mathrm{Q}_{\mathbf{x}_n} x_n \in \mathbf{x}_n') (\mathrm{Q}_{\mathbf{z}} z \in \mathbf{z}') (z = f(x_1, \ldots, x_n))$$

Similar to that negative numbers are the inverse elements of positive numbers in the real arithmetic, the introduction of improper intervals enables the Kaucher arithmetic to have group properties. Table 2 lists the major differences between MIA and IA. MIA offers better algebraic properties and more semantic capabilities.

Not only for outer range estimations, generalized intervals are also convenient for inner range estimations (Kupriyanova (1995); Kreinovich et al. (1996); Goldsztejn (2005)). For a solution set $\mathcal{S} \subset \mathbb{R}^n$ of the interval system $\mathbf{f}(\mathbf{x}) = 0$ where $\mathbf{x} \in \mathbb{IR}^n$, an inner estimation $\mathbf{x}^{in}$ of the solution set $\mathcal{S}$ is an interval vector that is guaranteed to be included in the solution set, and an outer estimation $\mathbf{x}^{out}$ of $\mathcal{S}$ is an interval vector that is guaranteed to include the solution set.

Our new interval probability representation incorporating the generalized interval is to take advantage of its algebraic properties so that the calculus of interval probability can be simplified. At the same time, the logic interpretation of probabilistic properties can be integrated so that the *completeness* and *soundness* of range estimates can be verified. A *complete* solution includes all possible occurrences, which is to check if the range estimation includes all possible combinations. Conversely, a *sound* solution does not include impossible occurrences, which consists in checking if the interval overestimates the actual range.

3 Imprecise Probability based on Generalized Intervals

3.1 Basic Concepts

Definition 3.1. *Given a sample space Ω and a σ-algebra $\mathcal{A}$ of random events over Ω, we define the generalized interval probability $\mathbf{p} \in \mathbb{KR}$ as $\mathbf{p} : \mathcal{A} \to [0, 1] \times [0, 1]$*

Table 2 The major differences between MIA and the traditional IA

	Classical Interval Analysis	**Modal Interval Analysis**
Validity	$[\underline{a},\overline{a}]$ with $\underline{a} > \overline{a}$ is invalid.	$\mathbf{a} = [\underline{a},\overline{a}]$ with $\underline{a} \le \overline{a}$ or $\underline{a} > \overline{a}$ are always valid.
Semantics	$[\underline{a},\overline{a}] + [\underline{b},\overline{b}] = [\underline{a}+\overline{a}, \underline{b}+\overline{b}]$, with $\underline{a} \le \overline{a}$ and $\underline{b} \le \overline{b}$, only means "stack-up" and "worst-case". $-, \times, \div$ are similar.	$\mathbf{a}+\mathbf{b}=\mathbf{c}$ has different meanings with different modalities. - when $\mathbf{a} \in \mathbb{IR}$, $\mathbf{b} \in \mathbb{IR}$, $\mathbf{c} \in \mathbb{IR}$: $\forall a \in \mathbf{a}', \forall b \in \mathbf{b}', \exists c \in \mathbf{c}', a+b=c$ - when $\mathbf{a} \in \mathbb{IR}$, $\mathbf{b} \in \overline{\mathbb{IR}}$, $\mathbf{c} \in \mathbb{IR}$: $\forall a \in \mathbf{a}', \exists b \in \mathbf{b}', \exists c \in \mathbf{c}', a+b=c$ - when $\mathbf{a} \in \mathbb{IR}$, $\mathbf{b} \in \overline{\mathbb{IR}}$, $\mathbf{c} \in \overline{\mathbb{IR}}$: $\forall a \in \mathbf{a}', \forall c \in \mathbf{c}', \exists b \in \mathbf{b}', a+b=c$ - when $\mathbf{a} \in \overline{\mathbb{IR}}$, $\mathbf{b} \in \overline{\mathbb{IR}}$, $\mathbf{c} \in \overline{\mathbb{IR}}$: $\forall c \in \mathbf{c}', \exists a \in \mathbf{a}', \exists b \in \mathbf{b}', a+b=c$ $-, \times, \div$ are similar.
Group property	- When $[\underline{a},\overline{a}]$ is a non-pointwise interval (i.e. $\underline{a} < \overline{a}$), $[\underline{a},\overline{a}] - [\underline{a},\overline{a}] \ne 0$, and $[\underline{a},\overline{a}] \div [\underline{a},\overline{a}] \ne 1$ for $0 \notin [\underline{a},\overline{a}]$. - If $[\underline{a},\overline{a}]$ and $[\underline{b},\overline{b}]$ are both non-pointwise intervals, then $[\underline{a},\overline{a}] + \mathbf{x} = [\underline{b},\overline{b}]$, but $\mathbf{x} \ne [\underline{b},\overline{b}] - [\underline{a},\overline{a}]$. Ex. $[2,3]+[2,4]=[4,7]$, but $[2,4] \ne [4,7]-[2,3]$. - The same for the case of $[\underline{a},\overline{a}] \times \mathbf{x} = [\underline{b},\overline{b}]$ for $(0 \notin [\underline{a},\overline{a}])$ that it holds but $\mathbf{x} \ne [\underline{b},\overline{b}] \div [\underline{a},\overline{a}]$ whenever $[\underline{a},\overline{a}]$ and $[\underline{b},\overline{b}]$ are both non-pointwise intervals.	$\mathbf{a} - \mathrm{dual}\mathbf{a} = 0$. $\mathbf{a} \div \mathrm{dual}\mathbf{a} = 1$ for $0 \notin \mathbf{a}'$. $\mathbf{a}+\mathbf{x}=\mathbf{b}$, and $\mathbf{x} = \mathbf{b} - \mathrm{dual}\mathbf{a}$. Ex. $[2,3]+[2,4]=[4,7]$, and $[2,4]=[4,7]-[3,2]$. $\mathbf{a} \times \mathbf{x} = \mathbf{b}$, and $\mathbf{x} = \mathbf{b} \div \mathrm{dual}\mathbf{a}$. Ex. $[2,3] \times [3,4] = [6,12]$, and $[3,4] = [6,12] \div [3,2]$.

which obeys the axioms of Kolmogorov: (1) $\mathbf{p}(\Omega) = [1,1]$*; (2)* $[0,0] \leq \mathbf{p}(E) \leq [1,1]$ $(\forall E \in \mathcal{A})$*; and (3) for any countable mutually disjoint events* $E_i \cap E_j = \emptyset$ $(i \neq j)$*,* $\mathbf{p}\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mathbf{p}(E_i)$*.*

Therefore, an interval probability $\mathbf{p} = \left[\underline{p}, \overline{p}\right]$ is a generalized interval without the restriction of $\underline{p} \leq \overline{p}$. The new definition of interval probability also implies $\mathbf{p}(\emptyset) = [0,0]$.

Definition 3.2. *The probability of the union of two events* E_1 *and* E_2 *is defined as*

$$\mathbf{p}(E_1 \cup E_2) := \mathbf{p}(E_1) + \mathbf{p}(E_2) - \mathrm{dual}\mathbf{p}(E_1 \cap E_2) \tag{4}$$

When the probabilities of E_1 and E_2 become precise, Eq.(4) has the same form as the traditional precise probabilities. From Eq.(4), we have

$$\mathbf{p}(E_1 \cup E_2) + \mathbf{p}(E_1 \cap E_2) = \mathbf{p}(E_1) + \mathbf{p}(E_2) \tag{5}$$

which also indicates the generalized interval probabilities are 2-monotone and 2-alternating in the sense of Choquet's capacities. For all $E_1, E_2 \in \mathcal{A}$, the lower probability $\underline{p}$ is said to be *2-monotone* if $\underline{p}(E_1 \cup E_2) + \underline{p}(E_1 \cap E_2) \geq \underline{p}(E_1) + \underline{p}(E_2)$, and the upper probability $\overline{p}$ is said to be *2-alternating* if $\overline{p}(E_1 \cup E_2) + \overline{p}(E_1 \cap E_2) \leq \overline{p}(E_1) + \overline{p}(E_2)$. However the relation in Eq.(5) is stronger than the 2-monotonicity.

Let $(\Omega, \mathcal{A})$ be the probability space and $\mathcal{P}$ a non-empty set of probability distributions on that space. The lower and upper probability envelopes are usually defined as

$$P_*(E) = \inf_{P \in \mathcal{P}} P(E)$$
$$P^*(E) = \sup_{P \in \mathcal{P}} P(E)$$

Not every probability envelope is 2-monotone. However, 2-monotone closed-form representations are more applicable because it may be difficult to track probability envelopes during manipulations. Therefore it is of our interest that a simple algebraic structure can provide such practical advantages for broader applications.

Furthermore, we have

$$\mathbf{p}(E_1 \cup E_2) \leq \mathbf{p}(E_1) + \mathbf{p}(E_2) \ (\forall E_1, E_2 \in \mathcal{A}) \tag{6}$$

in the new interval representation, since $\mathbf{p}(E_1 \cap E_2) \geq 0$. Note that Eq.(6) is different from the relation defined in the Dempster-Shafer structure or F-probability, where $\underline{p}(E_1) + \underline{p}(E_2) \leq \underline{p}(E_1 \cup E_2) \leq \underline{p}(E_1) + \overline{p}(E_2) \leq \overline{p}(E_1 \cup E_2) \leq \overline{p}(E_1) + \overline{p}(E_2)$. In Eq.(6), both lower and upper probabilities are subadditive. It has the same form as the precise probability except for the newly defined inequality $(\leq, \geq)$ relationships as in Eq.(2) for generalized intervals. Similar to the precise probability, the equality in Eq.(6) occurs when $\mathbf{p}(E_1 \cap E_2) = [0,0]$.

The values of interval probabilities are between 0 and 1. As a result, the interval probabilities $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ have the following algebraic properties:

$$\begin{aligned}\mathbf{p}_1 \leq \mathbf{p}_2 &\Leftrightarrow \mathbf{p}_1 + \mathbf{p}_3 \leq \mathbf{p}_2 + \mathbf{p}_3 \\ \mathbf{p}_1 \subseteq \mathbf{p}_2 &\Leftrightarrow \mathbf{p}_1 + \mathbf{p}_3 \subseteq \mathbf{p}_2 + \mathbf{p}_3 \\ \mathbf{p}_1 \leq \mathbf{p}_2 &\Leftrightarrow \mathbf{p}_1\mathbf{p}_3 \leq \mathbf{p}_2\mathbf{p}_3 \\ \mathbf{p}_1 \subseteq \mathbf{p}_2 &\Leftrightarrow \mathbf{p}_1\mathbf{p}_3 \subseteq \mathbf{p}_2\mathbf{p}_3\end{aligned}$$

Definition 3.3. *The probability of the complement of event E is*

$$\mathbf{p}(E^c) := 1 - \mathrm{dual}\mathbf{p}(E) \tag{7}$$

which is equivalent to

$$\underline{p}(E^c) := 1 - \overline{p}(E) \tag{8}$$

$$\overline{p}(E^c) := 1 - \underline{p}(E) \tag{9}$$

The definitions in Eqs.(8) and (9) are equivalent to the other forms of interval probabilities. The calculation based on generalized intervals as in Eq.(7) can be more concise. That is,

$$\mathbf{p}(E) + \mathbf{p}(E^c) = 1 \ (\forall E \in \mathcal{A}) \tag{10}$$

Eq.(10) can be generalized to a *logic coherence constraint* as follows.

Definition 3.4. *(Logic Coherence Constraint) For a mutually disjoint event partition $\bigcup_{i=1}^{n} E_i = \Omega$,*

$$\sum_{i=1}^{n} \mathbf{p}(E_i) = 1 \tag{11}$$

The logic coherence constraint is more restrictive than the traditional coherence constraint (Walley (1991)). Suppose that $\mathbf{p}(E_i) \in \mathbb{IR}$ (for $i = 1, \ldots, k$) and $\mathbf{p}(E_i) \in \overline{\mathbb{IR}}$ (for $i = k+1, \ldots, n$). Eq.(11) can be interpreted as

$$\forall p_1 \in \mathbf{p}'(E_1), \ldots, \forall p_k \in \mathbf{p}'(E_k), \exists p_{k+1} \in \mathbf{p}'(E_{k+1}), \ldots, \exists p_n \in \mathbf{p}'(E_n), \sum_{i=1}^{n} p_i = 1 \tag{12}$$

based on the interpretability principles of MIA (Gardeñes et al. (2001)).

Example 3.5. Given three events E_1, E_2, and E_3 in the sample space, $E_1 \cup E_2 \cup E_3 = \Omega$, and $E_3 = (E_1 \cup E_2)^c$. We know $\mathbf{p}(E_1) = [0.2, 0.3]$, $\mathbf{p}(E_2) = [0.4, 0.6]$, $\mathbf{p}(E_1 \cap E_2) = [0.1, 0.2]$. Then $\mathbf{p}(E_1 \cup E_2) = \mathbf{p}(E_1) + \mathbf{p}(E_2) - \mathrm{dual}\mathbf{p}(E_1 \cap E_2) = [0.2, 0.3] + [0.4, 0.6] - [0.2, 0.1] = [0.5, 0.7]$. Applying the logic coherence constraint $\mathbf{p}(E_1 \cup E_2) + \mathbf{p}(E_3) = 1$, we have $\mathbf{p}(E_3) = 1 - \mathrm{dual}\mathbf{p}(E_1 \cup E_2) = 1 - [0.7, 0.5] = [0.5, 0.3]$. The logic interpretation of the algebraic relation is

$$\forall p_{1,2} \in [0.5, 0.7]', \exists p_3 \in [0.3, 0.5]', p_{1,2} + p_3 = 1$$

A generalization of Eq.(4) is

$$\mathbf{p}(A) = \sum_{S \subseteq A} (-\mathrm{dual})^{|A|-|S|}\mathbf{p}(S) \tag{13}$$

for $A \subseteq \Omega$. For instance, for three events $E_i (i = 1, 2, 3)$,

$$\begin{aligned}\mathbf{p}(E_1 \cup E_2 \cup E_3) = \mathbf{p}(E_1) + \mathbf{p}(E_2) + \mathbf{p}(E_3) - \mathrm{dual}\mathbf{p}(E_1 \cap E_2)\\ -\mathrm{dual}\mathbf{p}(E_2 \cap E_3) - \mathrm{dual}\mathbf{p}(E_1 \cap E_3) + \mathbf{p}(E_1 \cap E_2 \cap E_3)\end{aligned}$$

The lower and upper probabilities in the generalized interval form do not just have the traditional meanings of lower and upper envelopes. They maintain the semantics of relationships among probability estimates as well as the associated events. Rather than only capturing the relationships among interval probabilities as in the traditional coherence constraint, the logic coherence constraint also incorporates events, which is the differentiation between focal and non-focal events.

3.2 Focal and Non-Focal Events

Definition 3.6. *An event E is a* ***focal*** *event if the associated semantics for $\mathbf{p}(E)$ is universal ($\mathrm{Q}_{\mathbf{p}(E)} = \forall$). Otherwise it is a* ***non-focal*** *event if the semantics is existential ($\mathrm{Q}_{\mathbf{p}(E)} = \exists$).*

Remark 3.7. Notice that the focal event is a different concept from focal sets defined in random sets.

A focal event is an event of interest in probabilistic analysis. The uncertainties associated with focal events are critical for the analysis of a system. In contrast, the uncertainties associated with non-focal events are “complementary” and “balancing”. The corresponding non-focal event is not the focus of the assessment. The quantified uncertainties of non-focal events are derived from those of the corresponding focal events. For instance, in risk assessment, the high-consequence event of interest is the target and focus of study, such as the event of a structural failure at the half of a bridge’s life expectancy, whereas the event of the structural failure when the bridge is twice as old as it was designed for may become non-focal.

In the interpretation in Eq.(12), the interval probability of a focal event E_i is proper ($\mathbf{p}(E_i) \in \mathbb{IR}$), and the interval probability of a non-focal event E_j is improper ($\mathbf{p}(E_j) \in \overline{\mathbb{IR}}$). Focal events have the semantics of *critical*, *uncontrollable*, *specified* in probabilistic analysis, whereas the corresponding non-focal events are *complementary*, *controllable*, and *derived*. The complement of a focal event is a non-focal event. For a set of mutually disjoint events, there is at least one non-focal event because of Eq.(11).

Two relationships between events are defined as follows.

Definition 3.8. *Event E_1 is said to be* ***less likely*** *(or* ***more likely****) to occur than event E_2, denoted as $E_1 \preceq E_2$ (or $E_1 \succeq E_2$), which is defined as*

$$\begin{aligned} E_1 \preceq E_2 &\iff \mathbf{p}(E_1) \leq \mathbf{p}(E_2) \\ (E_1 \succeq E_2 &\iff \mathbf{p}(E_1) \geq \mathbf{p}(E_2)) \end{aligned} \tag{14}$$

where $\leq$ is defined in Eq.(2).

Definition 3.9. *Event E_1 is said to be **less focused** (or **more focused**) than event E_2, denoted as $E_1 \sqsubseteq E_2$ (or $E_1 \sqsupseteq E_2$), which is defined as*

$$\begin{array}{c} E_1 \sqsubseteq E_2 \iff \mathbf{p}(E_1) \subseteq \mathbf{p}(E_2) \\ (E_1 \sqsupseteq E_2 \iff \mathbf{p}(E_1) \supseteq \mathbf{p}(E_2)) \end{array} \tag{15}$$

where $\subseteq$ is defined in Eq.(1).

With the above two relationships, the degree of imprecise belief and the level of imprecision are comparable. During analysis, usually we would like to concentrate on those events that are more focused first. For two focal events E_1 and E_2 which both have proper interval probabilities ($\mathbf{p}(E_1) \in \mathbb{IR}, \mathbf{p}(E_2) \in \mathbb{IR}$), when $\mathbf{p}(E_1) \subseteq \mathbf{p}(E_2)$, the width of the interval $\mathbf{p}(E_2)$ is greater than that of $\mathbf{p}(E_1)$. Thus E_2 has a higher level of uncertainty than E_1. For a non-focal event E_1 and a focal event E_2 ($\mathbf{p}(E_1) \in \overline{\mathbb{IR}}, \mathbf{p}(E_2) \in \mathbb{IR}$), when $\mathbf{p}(E_1) \subseteq \mathbf{p}(E_2)$, obviously the focal event E_2 is of interest to us.

Lemma 3.10. *(Monotonicity) $E_1 \subseteq E_2 \Rightarrow E_1 \preceq E_2$.*

Proof. $E_1 \subseteq E_2 \Rightarrow \mathbf{p}(E_2) = \mathbf{p}(E_1 \cup (E_2 - E_1)) = \mathbf{p}(E_1) + \mathbf{p}(E_2 - E_1) - \mathrm{dual}\mathbf{p}(E_1 \cap (E_2 - E_1)) \geq \mathbf{p}(E_1)$. □

Remark 3.11. A subset of events is less likely to occur than its superset.

Lemma 3.12. *(Additivity) If $E_1 \cap E_3 = \emptyset$ and $E_2 \cap E_3 = \emptyset$, then $E_1 \preceq E_2 \Leftrightarrow E_1 \cup E_3 \preceq E_2 \cup E_3$, $E_1 \sqsubseteq E_2 \Leftrightarrow E_1 \cup E_3 \sqsubseteq E_2 \cup E_3$.*

Proof. $E_1 \preceq E_2 \Leftrightarrow \mathbf{p}(E_1) \leq \mathbf{p}(E_2) \Leftrightarrow \mathbf{p}(E_1) + \mathbf{p}(E_3) \leq \mathbf{p}(E_2) + \mathbf{p}(E_3) \Leftrightarrow \mathbf{p}(E_1 \cup E_3) \leq \mathbf{p}(E_2 \cup E_3) \Leftrightarrow E_1 \cup E_3 \preceq E_2 \cup E_3$.

$E_1 \sqsubseteq E_2 \Leftrightarrow \mathbf{p}(E_1) \subseteq \mathbf{p}(E_2) \Leftrightarrow \mathbf{p}(E_1) + \mathbf{p}(E_3) \subseteq \mathbf{p}(E_2) + \mathbf{p}(E_3) \Leftrightarrow \mathbf{p}(E_1 \cup E_3) \subseteq \mathbf{p}(E_2 \cup E_3) \Leftrightarrow E_1 \cup E_3 \sqsubseteq E_2 \cup E_3$. □

Lemma 3.13. *If E_1 and E_3 are independent, and also E_2 and E_3 are independent, then $E_1 \preceq E_2 \Leftrightarrow E_1 \cap E_3 \preceq E_2 \cap E_3$, $E_1 \sqsubseteq E_2 \Leftrightarrow E_1 \cap E_3 \sqsubseteq E_2 \cap E_3$.*

Proof. $E_1 \preceq E_2 \Leftrightarrow \mathbf{p}(E_1) \leq \mathbf{p}(E_2) \Leftrightarrow \mathbf{p}(E_1)\,\mathbf{p}(E_3) \leq \mathbf{p}(E_2)\,\mathbf{p}(E_3) \Leftrightarrow \mathbf{p}(E_1 \cap E_3) \leq \mathbf{p}(E_2 \cap E_3) \Leftrightarrow E_1 \cap E_3 \preceq E_2 \cap E_3$.

$E_1 \sqsubseteq E_2 \Leftrightarrow \mathbf{p}(E_1) \subseteq \mathbf{p}(E_2) \Leftrightarrow \mathbf{p}(E_1)\,\mathbf{p}(E_3) \subseteq \mathbf{p}(E_2)\,\mathbf{p}(E_3) \Leftrightarrow \mathbf{p}(E_1 \cap E_3) \subseteq \mathbf{p}(E_2 \cap E_3) \Leftrightarrow E_1 \cap E_3 \sqsubseteq E_2 \cap E_3$. □

Lemma 3.14. *Suppose $\mathbf{p}(E) \in \mathbb{IR}$. (1) $\mathbf{p}(E) \leq \mathbf{p}(E^c)$ if $\overline{p}(E) \leq 0.5$; (2) $\mathbf{p}(E) \geq \mathbf{p}(E^c)$ if $\underline{p}(E) \geq 0.5$; (3) $\mathbf{p}(E) \supseteq \mathbf{p}(E^c)$ if $\underline{p}(E) \leq 0.5$ and $\overline{p}(E) \geq 0.5$.*

Proof. (1) Because $\mathbf{p}(E) \in \mathbb{IR}$, $\mathbf{p}(E^c) \in \overline{\mathbb{IR}}$, and $\mathbf{p}(E) + \mathbf{p}(E^c) = 1$, it is easy to see $\underline{p}(E) \leq \underline{p}(E^c)$ and $\overline{p}(E) \leq \overline{p}(E^c)$ if $\overline{p}(E) \leq 0.5$. (2) can be verified similarly. (3) If $\underline{p}(E) \leq 0.5$ and $\overline{p}(E) \geq 0.5$, then $\underline{p}(E^c) \geq 0.5$ and $\overline{p}(E^c) \leq 0.5$. Thus $\underline{p}(E) \leq \underline{p}(E^c)$ and $\overline{p}(E) \geq \overline{p}(E^c)$. □

Remark 3.15. A focal event E ($\mathbf{p}(E) \in \mathbb{IR}$) is less likely to occur than its complement if $\mathbf{p}(E) \leq 0.5$; E is more likely to occur than its complement if $\mathbf{p}(E) \geq 0.5$; otherwise, E is more focused than its complement. When E is a non-focal event, its complement E^c is a focal event. The relationships between $\mathbf{p}(E)$ and $\mathbf{p}(E^c)$ are just opposite.

The relationships of events defined in the above lemmata provide the basis of interpretation for our new interval probability structure. It is shown that proper and improper interval probabilities have corresponding physical meanings. Furthermore, interval probabilities based on generalized intervals have some similar properties such as monotonicity and additivity as precise probabilities. This is helpful to build an intuitive assessment framework.

3.3 Conditional Interval Probabilities

Different from the coherent provision or F-probability theories, we define conditional generalized interval probabilities based on marginal probabilities.

Definition 3.16. *The conditional interval probability* $\mathbf{p}(E|C)$ *for all* $E, C \in \mathcal{A}$ *is defined as*

$$\mathbf{p}(E|C) := \frac{\mathbf{p}(E \cap C)}{\mathrm{dual}\mathbf{p}(C)} = \left[\frac{\underline{p}(E \cap C)}{\underline{p}(C)}, \frac{\overline{p}(E \cap C)}{\overline{p}(C)}\right] \tag{16}$$

when $\mathbf{p}(C) > 0$.

Not only does the definition in Eq.(16) ensure the algebraic completion of the interval probability calculus, but also it is a generalization of the canonical conditional probability in F-probabilities. Different from the Dempster's rule of conditioning or geometric conditioning, this conditional structure maintains the algebraic relation between marginal and conditional probabilities. As a result,

$$\mathbf{p}(C|C) = 1 \tag{17}$$

Based on the interpretability principles of MIA (Gardeñes et al. (2001)), the interval relation $\mathbf{c} = \mathbf{a}/\mathbf{b}$ is interpreted as

$$\begin{cases} \forall a \in \mathbf{a}', \forall b \in \mathbf{b}', \exists c \in \mathbf{c}', c = a/b \text{ or} & (\text{when } \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{IR}) \\ \forall c \in \mathbf{c}', \exists a \in \mathbf{a}', \exists b \in \mathbf{b}', c = a/b & \\ \forall a \in \mathbf{a}', \exists b \in \mathbf{b}', \exists c \in \mathbf{c}', c = a/b \text{ or} & \left(\text{when } \mathbf{a}, \mathbf{c} \in \mathbb{IR}, \mathbf{b} \in \overline{\mathbb{IR}}\right) \\ \forall b \in \mathbf{b}', \forall c \in \mathbf{c}', \exists a \in \mathbf{a}', c = a/b & \\ \forall a \in \mathbf{a}', \forall c \in \mathbf{c}', \exists b \in \mathbf{b}', c = a/b \text{ or} & \left(\text{when } \mathbf{a} \in \mathbb{IR}, \mathbf{b}, \mathbf{c} \in \overline{\mathbb{IR}}\right) \\ \forall b \in \mathbf{b}', \exists a \in \mathbf{a}', \exists c \in \mathbf{c}', c = a/b & \end{cases}$$

Thus the available logic interpretations of the conditional interval probability in Eq.(16) are as follows.

- when $\mathbf{p}(E \cap C) \in \mathbb{IR}$, $\mathbf{p}(C) \in \overline{\mathbb{IR}}$, and $\mathbf{p}(E|C) \in \mathbb{IR}$

(18) $$\forall p_{E\cap C} \in \mathbf{p}'(E\cap C), \forall p_C \in \mathbf{p}'(C), \exists p_{E|C} \in \mathbf{p}'(E|C), p_{E|C} = \frac{p_{E\cap C}}{p_C}$$

or

(19) $$\forall p_{E|C} \in \mathbf{p}'(E|C), \exists p_{E\cap C} \in \mathbf{p}'(E\cap C), \exists p_C \in \mathbf{p}'(C), p_{E|C} = \frac{p_{E\cap C}}{p_C}$$

- when $\mathbf{p}(E\cap C) \in \mathbb{IR}$, $\mathbf{p}(C) \in \mathbb{IR}$, and $\mathbf{p}(E|C) \in \mathbb{IR}$

(20) $$\forall p_{E\cap C} \in \mathbf{p}'(E\cap C), \exists p_C \in \mathbf{p}'(C), \exists p_{E|C} \in \mathbf{p}'(E|C), p_{E|C} = \frac{p_{E\cap C}}{p_C}$$

or

(21) $$\forall p_{E|C} \in \mathbf{p}'(E|C), \forall p_C \in \mathbf{p}'(C), \exists p_{E\cap C} \in \mathbf{p}'(E\cap C), p_{E|C} = \frac{p_{E\cap C}}{p_C}$$

- when $\mathbf{p}(E\cap C) \in \mathbb{IR}$, $\mathbf{p}(C) \in \mathbb{IR}$, and $\mathbf{p}(E|C) \in \overline{\mathbb{IR}}$

(22) $$\forall p_{E\cap C} \in \mathbf{p}'(E\cap C), \forall p_{E|C} \in \mathbf{p}'(E|C), \exists p_C \in \mathbf{p}'(C), p_{E|C} = \frac{p_{E\cap C}}{p_C}$$

or

(23) $$\forall p_C \in \mathbf{p}'(C), \exists p_{E\cap C} \in \mathbf{p}'(E\cap C), \exists p_{E|C} \in \mathbf{p}'(E|C), p_{E|C} = \frac{p_{E\cap C}}{p_C}$$

The logic interpretations of interval conditional probabilities build the connection between point measurements and probability sets. Therefore, we may use them to check if a range estimation is a tight envelope. A tight envelope must be both complete and sound. We use the Example 3.1 in (Weichselberger (2000)) to illustrate.

Example 3.17. Given the following probabilities in the sample space $\Omega = E_1 \cup E_2 \cup E_3$,

$$\begin{array}{ll}
\mathbf{p}'(E_1) = [0.10, 0.25]' & \mathbf{p}'(E_2 \cup E_3) = [0.75, 0.90]' \\
\mathbf{p}'(E_2) = [0.20, 0.40]' & \mathbf{p}'(E_1 \cup E_3) = [0.60, 0.80]' \\
\mathbf{p}'(E_3) = [0.40, 0.60]' & \mathbf{p}'(E_1 \cup E_2) = [0.40, 0.60]'
\end{array}$$

A partition of Ω is $\mathcal{C} = \{C_1, C_2\}$, where $C_1 = E_1 \cup E_2$, $C_2 = E_3$, and $\mathbf{p}'(C_1) = [0.40, 0.60]'$.

Suppose $\mathbf{p}(E_1) = [0.10, 0.25]$ and $\mathbf{p}(C_1) = [0.60, 0.40]$, we have

$$\mathbf{p}(E_1|C_1) = \frac{[0.10, 0.25]}{[0.40, 0.60]} = [0.1666, 0.6250]$$

The interpretation

$$\forall p_{E_1} \in [0.10, 0.25]', \forall p_{C_1} \in [0.40, 0.60]', \exists p_{E_1|C_1} \in [0.1666, 0.6250]', p_{E_1|C_1} = \frac{p_{E_1}}{p_{C_1}}$$

indicates that the range estimation $[0.1666, 0.6250]'$ of $p\,(E_1|C_1)$ is *complete* in the sense that it considers all possible occurrences of $p(E_1)$ and $p(C_1)$. However, the range estimation is not necessarily a tight envelope.

On the other hand, if $\mathbf{p}(E_1) = [0.25, 0.10]$ and $\mathbf{p}(C_1) = [0.40, 0.60]$, we have

$$\mathbf{p}(E_1|C_1) = \frac{[0.25, 0.10]}{[0.60, 0.40]} = [0.6250, 0.1666]$$

The interpretation

$$\forall p_{E_1|C_1} \in [0.1666, 0.6250]', \exists p_{E_1} \in [0.10, 0.25]', \exists p_{C_1} \in [0.40, 0.60]', p_{E_1|C_1} = \frac{p_{E_1}}{p_{C_1}}$$

indicates that the range estimation $[0.1666, 0.6250]'$ is also *sound* in the sense that the range estimation is a tight envelope.

Suppose $\mathbf{p}(E_1) = [0.25, 0.10]$, $\mathbf{p}(E_2) = [0.20, 0.40]$, and $\mathbf{p}(C_1) = [0.60, 0.40]$, we have

$$\mathbf{p}(E_1|C_1) = \frac{[0.25, 0.10]}{[0.40, 0.60]} = [0.4166, 0.25]$$

$$\mathbf{p}(E_2|C_1) = \frac{[0.20, 0.40]}{[0.40, 0.60]} = [0.3333, 1.0]$$

The interpretations are

$$\forall p_{E_1|C_1} \in [0.25, 0.4166]', \forall p_{C_1} \in [0.40, 0.60]', \exists p_{E_1} \in [0.10, 0.25]', p_{E_1|C_1} = \frac{p_{E_1}}{p_{C_1}}$$
$$\forall p_{E_2} \in [0.20, 0.40]', \forall p_{C_1} \in [0.40, 0.60]', \exists p_{E_2|C_1} \in [0.3333, 1.0]', p_{E_2|C_1} = \frac{p_{E_2}}{p_{C_1}}$$

respectively. Combining the two, we can have the interpretation of

$$\begin{aligned} \forall p_{E_2} \in [0.20, 0.40]', \forall p_{C_1} \in [0.40, 0.60]', \forall p_{E_1|C_1} \in [0.25, 0.4166]', \\ \exists p_{E_1} \in [0.10, 0.25]', \exists p_{E_2|C_1} \in [0.3333, 1.0]', \\ p_{E_1|C_1} = \tfrac{p_{E_1}}{p_{C_1}}, p_{E_2|C_1} = \tfrac{p_{E_2}}{p_{C_1}} \end{aligned}$$

If events A and B are independent, then

$$\mathbf{p}(A|B) = \frac{\mathbf{p}(A)\mathbf{p}(B)}{\mathrm{dual}\mathbf{p}(B)} = \mathbf{p}(A) \tag{24}$$

For a mutually disjoint event partition $\bigcup_{i=1}^{n} E_i = \Omega$, we have

$$\mathbf{p}(A) = \sum_{i=1}^{n} \mathbf{p}(A|E_i)\mathbf{p}(E_i) \tag{25}$$

Theorem 3.18. *(Value of Contradictory Information) If $B \cap C = \emptyset$, then (1) $\mathbf{p}(A|C) \subseteq \mathbf{p}(A|B) \Leftrightarrow \mathbf{p}(A|B \cup C) \subseteq \mathbf{p}(A|B)$; (2) $\mathbf{p}(A|C) \subseteq \mathbf{p}(A|B) \Leftrightarrow \mathbf{p}(A|C) \subseteq \mathbf{p}(A|B \cup C)$; (3) $\mathbf{p}(A|C) \subseteq \mathbf{p}(A|B) \Leftrightarrow \mathbf{p}(A|C) \subseteq \mathbf{p}(A|B \cup C) \subseteq \mathbf{p}(A|B)$*

Proof. (1) $\mathbf{p}(A|C) \subseteq \mathbf{p}(A|B) \Leftrightarrow \mathbf{p}(A \cap C) / \mathrm{dual}\mathbf{p}(C) \subseteq \mathbf{p}(A|B) \Leftrightarrow \mathbf{p}(A \cap C) \subseteq \mathbf{p}(A|B)\mathbf{p}(C) \Leftrightarrow \mathbf{p}(A|B)\mathbf{p}(B) + \mathbf{p}(A \cap C) \subseteq \mathbf{p}(A|B)\mathbf{p}(B) + \mathbf{p}(A|B)\mathbf{p}(C) \Leftrightarrow \mathbf{p}(A \cap B) + \mathbf{p}(A \cap C) \subseteq \mathbf{p}(A|B)\mathbf{p}(B \cup C) \Leftrightarrow \mathbf{p}(A \cap (B \cup C)) \subseteq \mathbf{p}(A|B)\mathbf{p}(B \cup C) \Leftrightarrow \mathbf{p}(A \cap (B \cup C)) / \mathrm{dual}\mathbf{p}(B \cup C) \subseteq \mathbf{p}(A|B) \Leftrightarrow \mathbf{p}(A|B \cup C) \subseteq \mathbf{p}(A|B)$. (2) can be verified similarly. Combining (1) and (2), we receive (3).

□

Remark 3.19. The interpretation of the relationship (1) in Theorem 3.18, from left to right, is that if there are two pieces of evidence (B and C), and one (C) may provide more precise estimation about a focal event (A) than the other (B) may, then the new estimation of probability about the focal event (A) based on the disjunctively combined evidence can be more precise than the one based on only one of them (B), even though the two pieces of information are contradictory to each other. The other direction of the reasoning from right to left is that if the precision of the focal event estimation with the newly introduced evidence (C) is improved, the new evidence (C) must be more informative than the old one (B) although these two are contradictory.

Remark 3.20. The interpretation of the relationship (2) in Theorem 3.18, from left to right, is that if the precision of the focal event estimation with a contradictory evidence (B) is not improved compared to the old one with the evidence (C), then the new evidence ($B \cup C$) does not improve the precision on the estimation of the focal event (A). The other direction of the reasoning from right to left is that if the estimation about a focal event (A) becomes more precise if some new evidence (C) excludes some possibilities (B) from the original evidence ($B \cup C$), then the estimation of probability about the focal event (A) based on the new evidence (C) must be more precise than the one based on the excluded one (B) alone.

Remark 3.21. The relationship (3) in Theorem 3.18 indicates that a combination of two contradictory evidences achieves a compromised level of precisions between the two individuals alone.

3.4 Bayes' Rule with Generalized Intervals

The Bayes' rule with generalized intervals (GIBR) is defined as

$$\mathbf{p}(E_i|A) = \frac{\mathbf{p}(A|E_i)\mathbf{p}(E_i)}{\sum_{j=1}^{n} \mathrm{dual}\mathbf{p}(A|E_j)\mathrm{dual}\mathbf{p}(E_j)} \tag{26}$$

where $E_i (i = 1, \dots, n)$ are mutually disjoint event partitions of Ω and $\sum_{j=1}^{n} \mathbf{p}(E_j) = 1$.

The lower and upper probabilities in Eq.(26) are calculated as

$$\left[\underline{p}(E_i|A), \overline{p}(E_i|A)\right] = \left[\frac{\underline{p}(A|E_i)\underline{p}(E_i)}{\sum_{j=1}^{n} \underline{p}(A|E_j)\underline{p}(E_j)}, \frac{\overline{p}(A|E_i)\overline{p}(E_i)}{\sum_{j=1}^{n} \overline{p}(A|E_j)\overline{p}(E_j)}\right] \tag{27}$$

We can see that Eq.(26) is algebraically consistent with the conditional definition in Eq.(16), because

$$\begin{aligned} \sum_{j=1}^{n} \mathrm{dual}\mathbf{p}\left(A|E_j\right) \mathrm{dual}\mathbf{p}\left(E_j\right) &= \sum_{j=1}^{n} \mathrm{dual}\left[\mathbf{p}\left(A|E_j\right) \mathbf{p}\left(E_j\right)\right] \\ &= \mathrm{dual} \sum_{j=1}^{n} \mathbf{p}\left(A \cap E_j\right) \\ &= \mathrm{dual}\mathbf{p}\left(A\right) \end{aligned}$$

When $n = 2$, and let $\mathbf{p}(E) \in \mathbb{IR}$ and $\mathbf{p}(E^c) \in \overline{\mathbb{IR}}$, Eq.(26) becomes

$$\underline{p}(E|A) = \frac{\underline{p}(A|E)\underline{p}(E)}{\underline{p}(A|E)\underline{p}(E) + \underline{p}(A|E^c)\underline{p}(E^c)} = \frac{\underline{p}(A \cap E)}{\underline{p}(A \cap E) + \underline{p}(A \cap E^c)} \tag{28}$$

$$\overline{p}(E|A) = \frac{\overline{p}(A|E)\overline{p}(E)}{\overline{p}(A|E)\overline{p}(E) + \overline{p}(A|E^c)\overline{p}(E^c)} = \frac{\overline{p}(A \cap E)}{\overline{p}(A \cap E) + \overline{p}(A \cap E^c)} \tag{29}$$

When $\mathbf{p}(A \cap E) \in \mathbb{IR}$ and $\mathbf{p}(A \cap E^c) \in \overline{\mathbb{IR}}$, the above relation is equivalent to the well-known 2-monotone tight envelope, given as:

$$P_*(E|A) = \frac{P_*(A \cap E)}{P_*(A \cap E) + P^*(A \cap E^c)} \tag{30}$$

$$P^*(E|A) = \frac{P^*(A \cap E)}{P^*(A \cap E) + P_*(A \cap E^c)} \tag{31}$$

where P_* and P^* are the lower and upper probability bounds defined in the traditional interval probabilities. Here $P^*(A \cap E^c) = \underline{p}(A \cap E^c)$ and $P_*(A \cap E^c) = \overline{p}(A \cap E^c)$ are the estimations of the lower and upper probability envelopes.

Lemma 3.22. *(1)* $\mathbf{p}(A|E) \subseteq \mathbf{p}(A|E^c) \Leftrightarrow \mathbf{p}(E|A) \subseteq \mathbf{p}(E)$. *(2)* $\mathbf{p}(A|E) \supseteq \mathbf{p}(A|E^c) \Leftrightarrow \mathbf{p}(E|A) \supseteq \mathbf{p}(E)$.

Proof. (1)

$$\begin{aligned}
&\mathbf{p}(A|E) \subseteq \mathbf{p}(A|E^c)\\
\Leftrightarrow\ &\mathbf{p}(A \cap E)\,/\mathrm{dual}\mathbf{p}(E) \subseteq \mathbf{p}(A \cap E^c)\,/\mathrm{dual}\mathbf{p}(E^c)\\
\Leftrightarrow\ &\mathbf{p}(A \cap E)\,\mathbf{p}(E^c) \subseteq \mathbf{p}(A \cap E^c)\,\mathbf{p}(E)\\
\Leftrightarrow\ &\underline{p}(A \cap E)\,\underline{p}(E^c) \geq \underline{p}(A \cap E^c) \text{ and } \underline{p}(E)\,\overline{p}(A \cap E)\,\overline{p}(E^c) \leq \overline{p}(A \cap E^c)\,\overline{p}(E)\\
\Leftrightarrow\ &\underline{p}(A \cap E)\left[1 - \underline{p}(E)\right] \geq \underline{p}(A \cap E^c)\\
&\text{and } \underline{p}(E)\,\overline{p}(A \cap E)\left[1 - \overline{p}(E)\right] \leq \overline{p}(A \cap E^c)\,\overline{p}(E)\\
\Leftrightarrow\ &\underline{p}(A \cap E) \geq \underline{p}(A \cap E)\,\underline{p}(E) + \underline{p}(A \cap E^c)\\
&\text{and } \underline{p}(E)\,\overline{p}(A \cap E) \leq \overline{p}(A \cap E)\,\overline{p}(E) + \overline{p}(A \cap E^c)\,\overline{p}(E)\\
\Leftrightarrow\ &\mathbf{p}(A \cap E) \subseteq \mathbf{p}(A \cap E)\,\mathbf{p}(E) + \mathbf{p}(A \cap E^c)\,\mathbf{p}(E)\\
\Leftrightarrow\ &\mathbf{p}(A \cap E) \subseteq \left[\mathbf{p}(A \cap E) + \mathbf{p}(A \cap E^c)\right]\mathbf{p}(E)\\
\Leftrightarrow\ &\mathbf{p}(A \cap E)\,/\mathrm{dual}\left[\mathbf{p}(A \cap E) + \mathbf{p}(A \cap E^c)\right] \subseteq \mathbf{p}(E)\\
\Leftrightarrow\ &\mathbf{p}(E|A) \subseteq \mathbf{p}(E)
\end{aligned}$$

The proof of (2) is similar.

□

Remark 3.23. When the likelihood functions $\mathbf{p}(A|E)$ and $\mathbf{p}(A|E^c)$ as well as prior and posterior probabilities are proper intervals, we can interpret the above relation in Lemma 3.22 as follows. If the likelihood estimation of event A given event E occurs is more precise than that of event A given event E does not occur, then the extra information A can reduce the ambiguity of the prior estimation of E.

Lemma 3.24. *(1)* $\mathbf{p}(A|E) \geq \mathbf{p}(A|E^c) \Leftrightarrow \mathbf{p}(E|A) \geq \mathbf{p}(E)$. *(2)* $\mathbf{p}(A|E) \leq \mathbf{p}(A|E^c) \Leftrightarrow \mathbf{p}(E|A) \leq \mathbf{p}(E)$.

Proof. The proof is similar to the one for Lemma 3.22.

□

Remark 3.25. The interpretation of Lemma 3.24 is as follows. If the occurrence of event E increases the likelihood estimation of event A compared to the one without the occurrence of event E, then the extra information A will increase the probability of knowing that event E occurs.

Theorem 3.26. $\mathbf{p}(A|E) = \mathbf{p}(A|E^c) \Leftrightarrow \mathbf{p}(E|A) = \mathbf{p}(E)$.

Proof. We know $\mathbf{p}(A|E) = \mathbf{p}(A|E^c)$ is equivalent to $\mathbf{p}(A|E) \supseteq \mathbf{p}(A|E^c)$ and $\mathbf{p}(A|E) \subseteq \mathbf{p}(A|E^c)$. From Lemma 3.22, we have $\mathbf{p}(E|A) \supseteq \mathbf{p}(E)$ and $\mathbf{p}(E|A) \subseteq \mathbf{p}(E)$, thus $\mathbf{p}(E|A) = \mathbf{p}(E)$. Alternatively, $\mathbf{p}(A|E) = \mathbf{p}(A|E^c)$ is equivalent to $\mathbf{p}(A|E) \geq \mathbf{p}(A|E^c)$ and $\mathbf{p}(A|E) \leq \mathbf{p}(A|E^c)$. From Lemma 3.24, we have $\mathbf{p}(E|A) \geq \mathbf{p}(E)$ and $\mathbf{p}(E|A) \leq \mathbf{p}(E)$. Therefore, $\mathbf{p}(E|A) = \mathbf{p}(E)$.

□

Remark 3.27. The interpretation of the above theorem is as follows. The extra information A does not add much value to the assessment of event E if we have very similar likelihood ratios between $\mathbf{p}(A|E)$ and $\mathbf{p}(A|E^c)$.

Some examples of logic interpretations for the relationships between prior and posterior interval probabilities in Eq.(26) are as follows.

- when $\mathbf{p}(A|E_i) \in \mathbb{IR}$, $\mathbf{p}(E_i) \in \mathbb{IR}$, $\mathbf{p}(A|E_j) \in \mathbb{IR}\,(j = 1,\ldots,n,\ j \neq i)$, $\mathbf{p}(E_{j_1}) \in \mathbb{IR}\,(j_1 = 1,\ldots,k,\ j_1 \neq i)$, $\mathbf{p}(E_{j_2}) \in \overline{\mathbb{IR}}\,(j_2 = k+1,\ldots,n,\ j_2 \neq i)$ and $\mathbf{p}(E_i|A) \in \overline{\mathbb{IR}}$

$$\begin{aligned} &\forall_{j\neq i} p_{A|E_j} \in \mathbf{p}'(A|E_j), \forall_{j_1\neq i} p_{E_{j1}} \in \mathbf{p}'(E_{j_1}),\\ \exists p_{A|E_i} \in \mathbf{p}'(A|E_i), \exists p_{E_i} \in \mathbf{p}'(E_i), &\exists_{j_2\neq i} p_{E_{j2}} \in \mathbf{p}'(E_{j_2}), \exists p_{E_i|A} \in \mathbf{p}'(E_i|A),\\ &p_{E_i|A} = \frac{p_{A|E_i} p_{E_i}}{\sum_{j=1}^{n} p_{A|E_j} p_{E_j}} \end{aligned} \tag{32}$$

- when $\mathbf{p}(A|E_i) \in \overline{\mathbb{IR}}$, $\mathbf{p}(E_i) \in \overline{\mathbb{IR}}$, $\mathbf{p}(A|E_j) \in \overline{\mathbb{IR}}\,(j = 1,\ldots,n,\ j \neq i)$, $\mathbf{p}(E_j) \in \overline{\mathbb{IR}}\,(j = 1,\ldots,n,\ j \neq i)$, and $\mathbf{p}(E_i|A) \in \overline{\mathbb{IR}}$

$$\begin{aligned} \forall_{j\neq i} p_{A|E_j} \in \mathbf{p}'(A|E_j), \forall_{j\neq i} p_{E_j} \in \mathbf{p}'(E_j), &\forall p_{E_i|A} \in \mathbf{p}'(E_i|A),\\ &\exists p_{A|E_i} \in \mathbf{p}'(A|E_i), \exists p_{E_i} \in \mathbf{p}'(E_i),\\ &p_{E_i|A} = \frac{p_{A|E_i} p_{E_i}}{\sum_{j=1}^{n} p_{A|E_j} p_{E_j}} \end{aligned} \tag{33}$$

Notice that because both $\mathbf{p}(A|E_i)$ and dual$\mathbf{p}(A|E_i)$ occur in the GIBR of Eq.(26), the associated logic interpretation about $\mathbf{p}(A|E_i)$ is always existential. This indicates that the completeness of the posterior probability estimation $\mathbf{p}(E_i|A)$ cannot be checked by the interpretation itself. Yet the soundness of the posterior probability estimation can be checked by some interpretations such as the one in Eq.(33).

Example 3.28. Suppose we have a prior probability estimation $\mathbf{p}(E) = [0.3, 0.6]$ about an event E. Two pieces of evidences A and B may help to update the belief about E. If the likelihood functions are $\mathbf{p}(A|E) = [0.5, 0.9]$, $\mathbf{p}(A|E^c) = [0.2, 0.4]$, $\mathbf{p}(B|E) = [0.3, 0.8]$, and $\mathbf{p}(B|E^c) = [0.3, 0.8]$, then we have the posterior probabilities based on Eq.(26) as

$$\begin{aligned}\mathbf{p}(E|A) &= \frac{\mathbf{p}(A|E)\,\mathbf{p}(E)}{\mathrm{dual}\mathbf{p}(A|E)\,\mathrm{dual}\mathbf{p}(E) + \mathrm{dual}\mathbf{p}(A|E^c)\,\mathrm{dual}\mathbf{p}(E^c)}\\ &= \frac{[0.5, 0.9] \times [0.3, 0.6]}{[0.9, 0.5] \times [0.6, 0.3] + [0.4, 0.2] \times [0.4, 0.7]}\\ &= [0.5172, 0.7715]\end{aligned}$$

$$\begin{aligned}\mathbf{p}(E|B) &= \frac{\mathbf{p}(B|E)\,\mathbf{p}(E)}{\mathrm{dual}\mathbf{p}(B|E)\,\mathrm{dual}\mathbf{p}(E) + \mathrm{dual}\mathbf{p}(B|E^c)\,\mathrm{dual}\mathbf{p}(E^c)}\\ &= \frac{[0.3, 0.8] \times [0.3, 0.6]}{[0.8, 0.3] \times [0.6, 0.3] + [0.8, 0.3] \times [0.4, 0.7]}\\ &= [0.3, 0.6]\end{aligned}$$

With the evidence A, the imprecision about event E is reduced. However, the evidence B does not help to gain more information, since $\mathbf{p}(B|E) = \mathbf{p}(B|E^c)$.

4 Imprecise Dirichlet Model under the Logic Coherence Constraint

Statistical data in many applications including reliability assessment are often multinomial, that is, observations fall into two or more unordered categories. For example, components or systems may either *pass* or *fail* in accelerated life tests. The failures occur in J different time periods $T_1, \ldots, T_J$ for the components. They fail with K different failure modes $M_1, \ldots, M_K$. All of these observations are multinomial. The imprecise Dirichlet model (IDM) (Walley (1996b); Bernard (2005)) is for objective statistical inference from multinomial data with unknown chances. It models prior ignorance and does not rely on the assumptions of fixed categories. In this section, we extend the IDM and incorporate the new interval probability structure, which can be applied in reliability assessment. For instance, given partial information about the prior distributions of frequencies for different types of failure modes, which are imprecise, we conduct some experimental tests to update the prior estimations thus reducing the level of imprecision about frequencies.

Consider an infinite population of units, which are categorized in K categories or types. The proportion of units for K categories is characterized by the parameter $\theta = (\theta_1, \ldots, \theta_K)$, where $\theta_k \geq 0$ for all $k = 1, \ldots, K$ and $\sum_{k=1}^{K} \theta_k = 1$. The unknown parameter θ measures the chances of falling into the different categories. In an experiment, we test N sample units and receive different numbers of units in K categories, summarized by the counts $\mathbf{n} = (n_1, \ldots, n_K)$, where n_k is the observed number of units of type k and $\sum_{k=1}^{K} n_k = N$. The probability of observing $\mathbf{n}$ given the multinomial distribution with the parameter θ is $P(\mathbf{n}|\theta) = \binom{N}{\mathbf{n}} \theta_1^{n_1} \cdots \theta_K^{n_K}$, where $\binom{N}{\mathbf{n}}$ is the multinomial coefficient. In other words, the likelihood of the observation $\mathbf{n}$ given the parameter θ is proportional to $\theta_1^{n_1} \cdots \theta_K^{n_K}$, i.e.,

$$L(\theta|\mathbf{n}) \propto \prod_{k=1}^{K} \theta_k^{n_k} \tag{34}$$

The precise prior Dirichlet distribution $Dirichlet(s, \mathbf{t})$ for the parameter θ, where $\mathbf{t} = (t_1, \ldots, t_K)$, has the probability density function that is proportional to $\theta_1^{st_1-1} \cdots \theta_K^{st_K-1}$, i.e.,

$$\pi(\theta) \propto \prod_{k=1}^{K} \theta_k^{st_k-1} \tag{35}$$

where $s > 0$, $0 < t_k < 1$ for $k = 1, \ldots, K$, and $\sum_{k=1}^{K} t_k = 1$. Here t_k is the *prior frequency*, which is the mean of θ_k under the Dirichlet prior. The positive constant s is the *total prior strength*, which determines the influence of the prior distribution on posterior probabilities and how quickly the posterior probabilities converge as the statistical data accumulate. s is usually fixed and not depending on Ω in a defined model.

The density function of the precise posterior Dirichlet distribution $Dirichlet(N+s, \mathbf{t}^*)$, where $\mathbf{t}^* = (t_1^*, \ldots, t_K^*)$, is generated by multiplying the prior density function in Eq.(35) by the likelihood in Eq.(34) as

$$\pi(\theta|\mathbf{n}) \propto \prod_{k=1}^{K} \theta_k^{n_k+st_k-1} \tag{36}$$

where $t_k^* = (n_k + st_k)/(N + s)$.

If only partial information about the prior frequencies t_k's is available, the Dirichlet model becomes imprecise. Instead of near-ignorance prior as in the original IDM, which estimates the lower and upper bounds of Dirichlet posterior $\underline{t}_k^* = n_k/(N+s)$ and $\overline{t}_k^* = (n_k+s)/(N+s)$ with $\underline{t}_k = 0$ and $\overline{t}_k = 1$ respectively, we apply the logic coherence constraint on the prior estimates $\underline{t}_k$ and $\overline{t}_k$, that is,

$$\begin{cases} \sum_{k=1}^{K} \underline{t}_k = 1 \\ \sum_{k=1}^{K} \overline{t}_k = 1 \end{cases}$$

or

$$\sum_{k=1}^{K} \mathbf{t}_k = 1 \tag{37}$$

Then the Dirichlet posterior distribution about θ is estimated by

$$\mathbf{t}_k^* = \frac{n_k + s\mathbf{t}_k}{N + s} \tag{38}$$

The logic coherence constraint in Eq.(37) on the Dirichlet priors does not make the new imprecise Dirichlet model satisfy Walley's representation invariance principle (Walley (1996b)), which states that the posterior upper and lower probabilities assigned to an observable event should not depend on the refinement or coarsening of categories provided that the event remains unchanged. However, it does make sure the Dirichlet posteriors also follow the logic coherence constraint

$$\sum_{k=1}^{K} \mathbf{t}_k^* = 1 \tag{39}$$

given that s is precise.

Example 4.1. We consider the lifetime of a system T. The sample space is $\Omega = \{(0 \le T < a), (a \le T < 2a), (T \ge 2a)\}$ for some $a > 0$, that is, the lifetime can be categorized into three periods. Suppose that the respective priors are $\mathbf{t} = ([2/5, 3/5], [1/5, 3/10], [2/5, 1/10])$. The total prior strength $s = 2$ is a constant. We would like to estimate it from some life tests. The observations from the tests are $\mathbf{n} = (3, 5, 1)$. Then the Dirichlet posterior estimation based on Eq.(38) is

$$\begin{aligned}\mathbf{t}^* &= \left(\left[\frac{3+2\times\frac{2}{5}}{9+2}, \frac{3+2\times\frac{3}{5}}{9+2}\right], \left[\frac{5+2\times\frac{1}{5}}{9+2}, \frac{5+2\times\frac{3}{10}}{9+2}\right], \left[\frac{1+2\times\frac{2}{5}}{9+2}, \frac{1+2\times\frac{1}{10}}{9+2}\right]\right) \\ &= \left(\left[\frac{19}{55}, \frac{21}{55}\right], \left[\frac{27}{55}, \frac{28}{55}\right], \left[\frac{9}{55}, \frac{6}{55}\right]\right)\end{aligned}$$

The imprecision associated with $P(0 \le T < a)$ is reduced from $1/5$ to $2/55$ after the tests.

When we combine the sample space to $\Omega' = \{(0 \le T < a), (T \ge a)\}$, the respective priors become $\mathbf{t}' = ([2/5, 3/5], [3/5, 2/5])$ and the observations are $\mathbf{n} = (3, 6)$. The new Dirichlet posterior estimation becomes

$$\begin{aligned}\mathbf{t}'^* &= \left(\left[\frac{3+2\times\frac{2}{5}}{9+2}, \frac{3+2\times\frac{3}{5}}{9+2}\right], \left[\frac{6+2\times\frac{3}{5}}{9+2}, \frac{6+2\times\frac{2}{5}}{9+2}\right]\right) \\ &= \left(\left[\frac{19}{55}, \frac{21}{55}\right], \left[\frac{36}{55}, \frac{34}{55}\right]\right)\end{aligned}$$

Suppose that we have a near-ignorance prior about $P(0 \le T < a)$. The posterior estimation is the same as the one from the original IDM. The respective priors become $\mathbf{t}'' = ([0, 1-\epsilon], [1, \epsilon])$. With the observations $\mathbf{n} = (3, 6)$, the Dirichlet posterior estimation is

$$\begin{aligned}\mathbf{t}''^* &= \left(\left[\frac{3+2\times 0}{9+2}, \frac{3+2(1-\epsilon)}{9+2}\right], \left[\frac{6+2\times 1}{9+2}, \frac{6+2\epsilon}{9+2}\right]\right) \\ &= \left(\left[\frac{3}{11}, \frac{5-2\epsilon}{11}\right], \left[\frac{8}{11}, \frac{6+2\epsilon}{11}\right]\right) \\ &\approx \left(\left[\frac{3}{11}, \frac{5}{11}\right], \left[\frac{8}{11}, \frac{6}{11}\right]\right)\end{aligned}$$

One may notice that the imprecision of estimation $\mathbf{t}$ about the parameter θ is reduced faster in the new IDM than it is in the original one. This is due to the logic coherence constraint imposed on the prior estimates. In contrast, near-ignorance prior is always applied in the original IDM, which ensures that the representation invariance principle is satisfied.

5 Concluding Remarks

In this paper, we presented a new form of imprecise probability based on generalized intervals, which can be applied in reliability assessment. Generalized intervals allow the coexistence of proper and improper intervals. This enables the algebraic closure of arithmetic operations. The simplified probability calculus provides advantages for engineering applications.

We differentiate focal events from non-focal events by the modalities and semantics of interval probabilities. An event is focal when the semantics associated with its interval probability is universal, whereas it is non-focal when the semantics is existential. This differentiation allows us to have a simple and unified representation based on a logic coherence constraint.

The new conditional probabilities ensure the algebraic relation with marginal interval probabilities. And the new Bayes' updating rule is a generalization of the 2-monotone tight envelope updating rule under the new representation. Generalized intervals allow us to interpret the algebraic relations among intervals in terms of the first-order logic. This helps us to understand the relationship between individual measurements and probability sets as well as to check completeness and soundness of bounds.

The imprecise Dirichlet model under the logic coherence constraint exhibits the self-consistency for both priors and posteriors. So does the logic interpretation when the imprecision is only from the prior frequencies. When priors are not near-ignorance, imprecision in the new imprecise Dirichlet model is reduced faster than in the original one.

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