Safety, Reliability, Risk and Life-Cycle Performance of Structures & Infrastructures – Deodatis, Ellingwood & Frangopol (Eds)
© 2013 Taylor & Francis Group, London, ISBN 978-1-138-00086-5

Sensitivity analysis for quantified interval constraints

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ABSTRACT: Interval is an alternative to probability distribution in quantifying uncertainty for reliability analysis when the distribution is unavailable. It only requires the information of lower and upper bounds. The effect of uncertain parameters can be estimated by solving interval-valued Constraint Satisfaction Problems (CSPs). By introducing logic quantifiers, Quantified Constraint Satisfaction Problems (QCSPs) can capture more semantics and engineering intent than CSPs. Sensitivity analysis (SA) takes into account variations of the structure and parameters of interval constraints to study to which extent they affect the output. In this paper, a global SA method is developed for QCSPs, where the effects of quantifiers and interval ranges on the constraints are analyzed based on several proposed metrics, which indicate the levels of indeterminacy for inputs and outputs as well as unsatisfiability of constraints. Engineering design problems are used to demonstrate the proposed approach.

1 INTRODUCTION

Interval is an alternative to probability distribution in quantifying uncertainty when the distribution is unavailable. It has the practical advantage of only requiring the minimum information of lower and upper bounds without assuming any distribution. Hence, interval analysis becomes attractive in analyzing the reliability of engineering design. The reliability of an engineering design can be estimated by solving the constraints with the consideration of the variations of parameters. That is, the effect of uncertain parameters on the performances of the engineering system can be estimated by solving interval-valued constraint satisfaction problems (CSPs).

Interval-valued CSP is a system of constraints where the variables are interval values. Solving an interval-valued CSP is to find the interval values of variables that satisfy all constraints. In order to capture more semantics, a CSP can be extended to a quantified constraint satisfaction problem (QCSP) which allows for universal (∀) and existential (∃) quantifiers associated with variables (Börner et al., 2003, Shary, 2002). Those variables which are not controllable by designer can be associated with quantifier ∀. They usually correspond to the external disturbance of a system. The variables which can be controlled and modified within some prescribed interval ranges by designer are associated with quantifier ∃. Therefore, design intents are captured by assigning quantifiers to variables. QCSP is a general problem with its solution satisfying constraints in the form of both mathematical and logic expressions.

An interval-valued QCSP is typically solved by applying the consistency techniques numerically in the iterative solving process. It can be simplified by applying the semantic analysis based on the logic interpretation. However, when constraints are overly restrictive on their interval value ranges, we may not be able to find feasible solutions. Additionally, when all variables in a constraint are universally quantified and interpretation is not possible, the constraint is regarded as being over-constrained in logic. In these cases, we need to know which input may have more impact on the output than the others so that adjustments can be performed to receive desirable solutions, in addition to whether there is a feasible solution or not during the solving process. Therefore, sensitivity analysis for interval-valued QCSPs is necessary and useful to gain such information.

Sensitivity analysis (SA) is to determine the relative contribution of inputs to the observed variation of outputs in a mathematical model. In contrast to variance-based statistical sensitivity analysis, few efforts have been taken for interval-valued models in uncertainty quantification. Early research focused on applying interval analysis to rigorously bound the sensitivity estimation of real-valued models, where the outer and inner estimations of Jacobian for nonlinear equations are computed (Neumaier, 1989, Rump, 1990, Wallner et al., 2005, Goldsztejn, 2008). More recently, a local SA method was proposed to study the impacts of the width and midpoint value changes of interval variables (Guo and Du, 2009). A hybrid approach to analyze the sensitivity of interval variables in multi-objective optimization problems via statistical sampling was also proposed (Li and Williams, 2009, Li et al., 2010) in which the sensitivities of parameters with respect to the multiple outputs are measured by Shannon entropy. Different from the above, the SA approach proposed here is to analyze the sensitivity of interval inputs globally with respect to the solutions of QCSPs.

Sampling is widely used in the traditional SA. It aims to generate the profiles of outputs so that the impacts of inputs can be compared. Sampling implies high computational costs in which a large number of calculations are conducted repeatedly, especially for global sensitivity analysis (GSA) where the entire range of input variations needs to be studied. The existing GSA techniques are mainly variance-based (Homma and Saltelli, 1996, Wagner, 1995, Sobol′, 2001, Chen et al., 2005, Saltelli et al., 2008, Swiler et al., 2009, Weirs et al., 2012) where the sampling efficiency is also closely related to the distributions.

In this paper, a GSA approach is developed for an interval-valued QCSP without assuming the distributions for the inputs. It aims at analyzing the effects of both interval value and quantifier of an input on the output of a constraint. With this information, the input can be adjusted for feasible solution in applications. Generalized interval (Dimitrova et al., 1992, Kaucher, 1980, Gardeñes et al., 2001) is used to represent the quantified inputs. It provides the convenience for numerical calculations in interval analysis and logic interpretations in assessing enclosure.

We use a metric, *unsatisfiability* denoted by $u(\cdot)$, to qualitatively measure whether a quantified interval constraint is satisfiable or not. We also define a metric, *indeterminacy* denoted by $\mathfrak{M}(\cdot)$, as a generalization of the *Hartley like measure* (Klir, 2006) to measure the change of information from an output as a result of the input variation. The major extension is that $\mathfrak{M}(\cdot)$ for a proper interval is defined as positive but negative for an improper interval. Additionally, the indeterminacy of a generalized interval vector is defined as a vector instead of a scalar value.

The interval-valued QCSPs can be specified by a constraint system in the form of $F(\mathbf{a}, \mathbf{x}) = \mathbf{b}$ in which $\mathbf{a}$ is the input, $\mathbf{b}$ is the output, and $\mathbf{x}$ is the unknown variables. The main idea of the proposed GSA approach is summarized as follows. The sensitivity of every input with respect to each constraint is estimated both quantitatively and qualitatively. Three representative values of an interval inputs are chosen to be the references based on which the difference of indeterminacies $\mathfrak{M}(\cdot)$'s for an output are calculated. The three *representative values* are lower bound, midpoint, and upper bound of an interval input. The information gain is quantified by the difference between indeterminacies of outputs. In order to differentiate the impact of each input on an output, the GSA is implemented in a framework of making the variation of one input at a time. The interaction of two inputs is estimated by varying the two inputs simultaneously. The sensitivities of inputs are ranked by some *sensitivity zones* which are generated by computing the total information gains for the three different representative values of an input. When two inputs have the same total information gain, qualitative metrics $u(\cdot)$'s are compared.

In the remainder of the paper, the background of generalized interval and Hartley like measure is introduced in Section 2. The proposed approach and metrics are described in Section 3. Two engineering design examples are presented to demonstrate the proposed method in Section 4.

2 BACKGROUND

2.1 *Generalized interval*

Generalized interval is an algebraic and semantic extension of the classical interval (Moore et al., 2009). The classical interval is defined as a set of real numbers as $[\![\underline{x}, \overline{x}]\!] := \{x \in \mathbb{R} | \underline{x} \leq x \leq \overline{x}\}$. In contrast, a generalized interval $\mathbf{x} := [\underline{x}, \overline{x}] \in \mathbb{KR}$, defined by a pair of numbers, is no longer restricted to the ordered bounds $\underline{x} \leq \overline{x}$. An operator Δ maps a generalized interval $\mathbf{x}$ to a classical interval, defined as

$$[\underline{x}, \overline{x}]^{\Delta} := \begin{cases} [\![\underline{x}, \overline{x}]\!], \text{ if } \underline{x} \leq \overline{x} \\ [\![\overline{x}, \underline{x}]\!], \text{ if } \underline{x} \geq \overline{x} \end{cases} \tag{1}$$

$\mathbf{x}$ is *proper* when $\underline{x} \leq \overline{x}$ and denoted as $\mathbf{x} \in \mathbb{IR}$. $\mathbf{x}$ is *improper* when $\underline{x} \geq \overline{x}$ and denoted as $\mathbf{x} \in \overline{\mathbb{IR}}$. Pointwise interval $\mathbf{x}$, when $\underline{x} = \overline{x}$, can be either proper or improper. The property of proper or improper is referred to as the *modality* of the interval. Operators *pro* and *imp* return proper and improper intervals respectively and are defined as

$$\mathrm{pro}[\underline{x}, \overline{x}] := [\min(\underline{x}, \overline{x}), \max(\underline{x}, \overline{x})] \tag{2}$$

$$\mathrm{imp}[\underline{x}, \overline{x}] := [\max(\underline{x}, \overline{x}), \min(\underline{x}, \overline{x})] \tag{3}$$

The relationship between proper and improper intervals is established by an operator *dual*, defined as $\mathrm{dual}([\underline{x}, \overline{x}]) := [\overline{x}, \underline{x}]$. Functions $\inf([\underline{x}, \overline{x}]) = \underline{x}$ and $\sup([\underline{x}, \overline{x}]) = \overline{x}$ return the lower and upper bounds of $\mathbf{x}$, respectively.

The generalized interval arithmetic is also called Kaucher arithmetic, which coincides with classical interval arithmetic when only proper intervals are involved. The intersection between two generalized intervals $\mathbf{x}$ and $\mathbf{y}$ is generally defined as

$$[\underline{x}, \overline{x}] \cap [\underline{y}, \overline{y}] = [\max(\underline{x}, \underline{y}), \min(\overline{x}, \overline{y})] \tag{4}$$

which also holds for the intersection of classical intervals, originally defined as

$$[\![\underline{x}, \overline{x}]\!] \cap [\![\underline{y}, \overline{y}]\!] := \{r \mid r \in [\![\underline{x}, \overline{x}]\!] \wedge r \in [\![\underline{y}, \overline{y}]\!]\} \tag{5}$$

The *inclusion* relationship between $\mathbf{x}$ and $\mathbf{y}$ is defined as

$$[\underline{x}, \overline{x}] \subseteq [\underline{y}, \overline{y}] \Leftrightarrow \underline{y} \leq \underline{x} \wedge \overline{x} \leq \overline{y} \quad (6)$$

The inclusion monotonicity is a useful property of Kaucher arithmetic which is expressed as

$$\mathbf{y}_1 \subseteq \mathbf{x}_1, \mathbf{y}_2 \subseteq \mathbf{x}_2 \Rightarrow \mathbf{y}_1 \bullet \mathbf{x}_1 \subseteq \mathbf{y}_2 \bullet \mathbf{x}_2 \quad (7)$$

where $\bullet \in \{+,-,\times,/\}$, $\mathbf{x}_1$, $\mathbf{y}_1$, $\mathbf{x}_2$, and $\mathbf{y}_2$ are generalized intervals. It also states that if $F = F(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ is an interval extension of function f, which only involves the operations $\bullet \in \{+,-,\times,/\}$ and $\mathbf{y}_i \subseteq \mathbf{x}_i$, for $i = 1, \ldots, n$, then $F(\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n) \subseteq F(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$.

The *width* of a generalized interval $\mathbf{x} = [\underline{x}, \overline{x}]$ is defined by $\text{wid}(\mathbf{x}) := |\overline{x} - \underline{x}|$ which is a non-negative value to quantify the size of an interval. The midpoint value is found by $\text{mid}(\mathbf{x}) := (\overline{x} + \underline{x})/2$, and the *radius* is defined by $\text{rad}(\mathbf{x}) := (\overline{x} - \underline{x})/2$, which is positive when $\mathbf{x} \in \mathbb{KR}$ and negative when $\mathbf{x} \in \overline{\mathbb{KR}}$.

Generalized intervals based on Kaucher arithmetic form a *group*, whereas classical intervals form a *semi-group* without invertibility. Generalized interval also provides richer semantics than the classical interval. Given a constraint system of a QCSP

$$F(\mathbf{a}, \mathbf{x}) = \mathbf{b} \quad (8)$$

where $F : \mathbb{KR}^l \times \mathbb{KR}^n \rightarrow \mathbb{KR}^m$, the elements in $\mathbf{a}, \mathbf{x}$, and $\mathbf{b}$ are generalized intervals. With the notations ρ for proper and ι for improper, a proper interval vector $\mathbf{a}_\rho \in \mathbb{IR}^l$ with its i-th element $(\mathbf{a}\rho)_i$ and an improper interval vector $\mathbf{a}_\iota \in \overline{\mathbb{IR}}^l$ with its i-th element $(\mathbf{a}_\iota)_i$ are defined as

$$(\mathbf{a}_\rho)_i := \begin{cases} \mathbf{a}_i, \text{if } \underline{a}_i \leq \overline{a}_i \\ 0, \text{otherwise} \end{cases}, \quad (\mathbf{a}_\iota)_i := \begin{cases} \mathbf{a}_i, \text{if } \underline{a}_i \geq \overline{a}_i \\ 0, \text{otherwise} \end{cases} \quad (9)$$

such that $\mathbf{a} = \mathbf{a}\rho + \mathbf{a}\iota$. Similarly, $\mathbf{b} = \mathbf{b}_\rho + \mathbf{b}_\iota$ and $\mathbf{x} = \mathbf{x}_\rho + \mathbf{x}_\iota$ are defined. The solution set of Equation 8 is interpreted as either

$$\begin{aligned} &(\forall x_\rho \in \mathbf{x}_\rho^\Delta)(\forall a_\rho \in \mathbf{a}_\rho^\Delta)(\forall b_\iota \in \mathbf{b}_\iota^\Delta)(\exists x_\iota \in \mathbf{x}_\iota^\Delta) \\ &(\exists a_\iota \in \mathbf{a}_\iota^\Delta)(\exists b_\rho \in \mathbf{b}_\rho^\Delta) f(a_\rho + a_\iota, x_\rho + x_\iota) = b_\rho + b_\iota \end{aligned} \quad (10)$$

or

$$\begin{aligned} &(\forall x_\iota \in \mathbf{x}_\iota^\Delta)(\forall a_\iota \in \mathbf{a}_\iota^\Delta)(\forall b_\rho \in \mathbf{b}_\rho^\Delta)(\exists x_\rho \in \mathbf{x}_\rho^\Delta) \\ &(\exists a_\rho \in \mathbf{a}_\rho^\Delta)(\exists b_\iota \in \mathbf{b}_\iota^\Delta) f(a_\rho + a_\iota, x_\rho + x_\iota) = b_\rho + b_\iota \end{aligned} \quad (11)$$

2.2 *Hartley like measure*

Hartley like measure (Klir, 2006), denoted by $HL(\cdot)$, is defined in a bounded and convex subset of $\mathbb{R}^n$ for some $n \geq 1$. For the convex subsets A, the $HL(A)$ is defined as

$$HL(A) = \min_{t \in T}\{c \log_b[\prod_{i=1}^{n}[1 + \mu(A_{it})] + \mu(A) - \prod_{i=1}^{n}\mu(A_{it})]\} \quad (12)$$

where μ is the Lebesgue measure, T is the set of all isometric transformations from one orthogonal coordinate system to another, A_{it} is the ith projection of A in coordinate system t, b and c are positive constants ($b \neq 1$) whose values define a measurement unit.

$HL(A)$ measures the *nonspecificity* of a bounded and convex subset. Nonspecificity characterizes the uncertainty caused by the quantity of possible alternatives in the considered set of values. For a closed interval I which is a set of real numbers, $HL(I)$ can be simplified as

$$HL(I) = c \log_b(1 + \text{wid}(I)) \quad (13)$$

which satisfies some essential requirements (Klir, 2006, Rényi, 1970) for an uncertainty measure.

For pointwise interval $[\![\underline{r}, \overline{r}]\!]$ in which $\underline{r} = \overline{r}$, $HL(r)$ is zero, meaning that it is completely specified. The value of $HL(\cdot)$ depends on the width of interval r.

3 GLOBAL SENSITIVITY ANALYSIS APPROACH FOR QCSP

3.1 *Basic metrics*

The proposed sensitivity analysis of interval constraints is based on the variation of interval ranges. A *variation* of interval $\mathbf{x} = [\underline{x}, \overline{x}]$, denoted by v_{x}, is defined as a value change to $\mathbf{x}' = [\underline{x}', \overline{x}']$ with $\underline{x}' = \underline{x} + \delta_1$ and $\overline{x}' = \overline{x} - \delta_2$ where $0 < \delta_1 \leq 2\text{rad}(\mathbf{x})$ and $0 < \delta_2 \leq 2\text{rad}(\mathbf{x})$ if $\mathrm{x} \in \mathbb{IR}$, and $2\text{rad}(\mathbf{x}) \leq \delta_1 < 0$ and $2\text{rad}(\mathbf{x}) \leq \delta_2 < 0$ if $\mathrm{x} \in \overline{\mathbb{IR}}$.

When $\delta_1 < \text{rad}(\mathbf{x})$ and $\delta_2 < \text{rad}(\mathbf{x})$ for $\mathrm{x} \in \mathbb{IR}$, and $\delta_1 > \text{rad}(\mathbf{x})$ and $\delta_2 > \text{rad}(\mathbf{x})$ for $\mathrm{x} \in \overline{\mathbb{IR}}$, the variation is called *local*. When $\delta_1 \geq \text{rad}(\mathbf{x})$ and $\delta_2 \geq \text{rad}(\mathbf{x})$ for $\mathrm{x} \in \mathbb{IR}$, and $\delta_1 \leq \text{rad}(\mathbf{x})$ and $\delta_2 \leq \text{rad}(\mathbf{x})$ for $\mathrm{x} \in \overline{\mathbb{IR}}$, the variation is called *global*. There is a change of modality and the associated logic quantifier in global variation. For instance, if the interval [1,2] is changed to [1.1,1.9], the variation is local. If it is changed to [1.5,1.5] or [2,1], the variation is global.

Notice that the term, global, used in the traditional sensitivity analysis means output variances are evaluated within the entire ranges of all inputs. Here, global indicates that the variation causes the interval modality change, which is more significant than the value change within the input range.

Consider a constraint system with m constraints and l variables. The first metric for sensitivity analysis is *unsatifiability*, defined as

$$u_j(\mathbf{a}, \mathbf{x}) = \begin{cases} 1, \text{if } c_j \text{ is unsatisfiable} \\ 0, \text{otherwise} \end{cases} \quad (14)$$

which indicates whether the jth constraint c_j is unsatisfiable or not with variables $\mathbf{a}$ and $\mathbf{x}$. The *total unsatisfiability* of the constraint system with m constraints is defined as

$$U(\mathbf{a}, \mathbf{x}) = \sum_{j=1}^{m} u_j(\mathbf{a}, \mathbf{x}) \quad (15)$$

The second metric is the *indeterminacy measure*, denoted as $\mathfrak{M}(\cdot)$, which is a generalization of the Hartley like measure $HL(\cdot)$ with the consideration of generalized intervals. The indeterminacy measure for a generalized interval $\mathbf{x} \in \mathbb{KR}$ is defined as

$$\mathfrak{M}(\mathbf{x}) = \begin{cases} \log_2(1 + \mathrm{wid}(\mathbf{x})), \text{ if } \mathbf{x} \in \mathbb{IR} \\ -\log_2(1 + \mathrm{wid}(\mathbf{x})), \text{if } \mathbf{x} \in \overline{\mathbb{IR}} \end{cases} \quad (16)$$

A negative $\mathfrak{M}(\cdot)$ is associated with an improper interval so that the indeterminacy measures for two generalized intervals can be differentiated when they have equal widths but are associated with different quantifiers.

The indeterminacy of a generalized interval vector $\mathbf{x} \in \mathbb{KR}^l$ is defined as a vector $\mathfrak{M}(\mathbf{x}) = [\mathfrak{M}(\mathbf{x}_1), \ldots, \mathfrak{M}(\mathbf{x}_l)]$, which measures the indeterminacy of each element $\mathbf{x}_i$ in $\mathbf{x}$ separately.

3.2 *Sensitivity estimated by indeterminacy of output*

Given a system of constraints in Equation 8 where $\mathbf{x} \in \mathbb{KR}^n$ is unknown, $\mathbf{a} \in \mathbb{KR}^l$ is the input, and $\mathbf{b} \in \mathbb{KR}^m$ is the output, the indeterminacy $\mathfrak{M}(\mathbf{b})$ of $\mathbf{b}$ is called *initial indeterminacy* when all inputs of $\mathbf{a}$ take their initially given interval ranges. The *remaining indeterminacy* $\mathfrak{M}(\mathbf{b}|a_i)$ is defined as the indeterminacy of $\mathbf{b}$ given that one input $\mathbf{a}_i$ becomes known with certainty as a real value whereas others remain as intervals. That is,

$$\mathfrak{M}(\mathbf{b}_j \mid a_i) = \begin{cases} \log_2(1 + \mathrm{wid}(\boldsymbol{F}_j(\mathbf{a}_{\sim i}, a_i, \mathbf{x}))), \text{ if } \boldsymbol{F}_j(\mathbf{a}_{\sim i}, a_i, \mathbf{x}) \in \mathbb{IR} \\ -\log_2(1 + \mathrm{wid}(\boldsymbol{F}_j(\mathbf{a}_{\sim i}, a_i, \mathbf{x}))), \text{if } \boldsymbol{F}_j(\mathbf{a}_{\sim i}, a_i, \mathbf{x}) \in \overline{\mathbb{IR}} \end{cases} \quad (17)$$

where $\sim i$ denotes the rest of elements in a vector except the ith one. Here, the remaining indeterminacy of output is obtained by choosing a_i as a *representative value*, i.e. $\inf(\mathbf{a}_i)$, $\mathrm{mid}(\mathbf{a}_i)$, or $\sup(\mathbf{a}_i)$, while other inputs remaining the original interval values.

Similarly, the *joint remaining indeterminacy*, $\mathfrak{M}(\mathbf{b}|a_i, a_k)$, can be defined with $\mathbf{b} = \boldsymbol{F}(\mathbf{a}_{\sim i \sim k}, a_i, a_k, \mathbf{x})$ by choosing a_i and a_k as the same corresponding representative values of $\mathbf{a}_i$ and $\mathbf{a}_k$.

Thus, the *main information gain* by knowing $\mathbf{a}_i$ with certainty with respect to the jt constraint is quantified as

$$I_j^m(a_i) = [\mathfrak{M}(\mathbf{b}_j) - \mathfrak{M}(\mathbf{b}_j \mid a_i)] \,/\, \mathfrak{M}(\mathbf{b}_j) \quad (18)$$

We say $I_j^m(\cdot)$ is *computable* if

$$\mathfrak{M}(\mathbf{b}_j) \times \mathfrak{M}(\mathbf{b}_j \mid a_i) \geq 0 \quad (19)$$

and

$$\mathfrak{M}(\mathbf{b}_j) \neq 0 \quad (20)$$

The two computable conditions in Equations 19–20 ensure that the definition of $I_j^m(a_i)$ in Equation 18 holds. The computable condition of Equation 19 requires $\mathfrak{M}(\mathbf{b}_j)$ and $\mathfrak{M}(\mathbf{b}_j|a_i)$ have the same sign, which indicates that only the numerical value of indeterminacy of the jth output is changed with known a_i. $I_j^m(a_i)$ reveals the numerical change of indeterminacy of the jth output when $\mathbf{a}_i$ is changed to a_i.

$\mathfrak{M}(\mathbf{b}_j) \times \mathfrak{M}(\mathbf{b}_j|a_i) < 0$ implies that the quantifier of the jth output is also changed besides of its value change when $\mathbf{a}_i$ is changed to a_i. In this scenario, the input $\mathbf{a}_i$ introduces the additional indeterminacy to the constraint. The quantifier change of the jth output when $\mathfrak{M}(\mathbf{b}_j) \times \mathfrak{M}(\mathbf{b}_j|a_i) < 0$ is measured by the *quantifier mutation gain* defined as

$$Q_j(a_i) = \mathfrak{M}(\mathbf{b}_j \mid a_i) \,/\, \mathfrak{M}(\mathbf{a}_i) \quad (21)$$

where $\mathbf{b}_j = \boldsymbol{F}_j(\mathbf{a}_{\sim i}, a_i, \mathbf{x}), i = 1, \ldots, n,$ and $j = 1, \ldots, m$.

The second computable condition in Equation 20 requires that the denominator $\mathfrak{M}(\mathbf{b}_j)$ should be nonzero. When $\mathfrak{M}(\mathbf{b}_j) = 0$, the output b_j is a real number with zero interval width. It implies that the uncertain inputs compensate to each other so that the output becomes a precisely known value. Any change of the inputs may have influence on the output such that the output becomes uncertain again. Since the pointwise interval can be treated as either proper or improper, the quantifier can be seen as either changed or not when an interval width is reduced to zero. In this paper, we treat it as a change so that the indeterminacy of output in this scenario can also be quantified by Equation 21.

The indeterminacy of an output has two levels. One is the numerical change with the same quantifier, and the other is the quantifier change. With the same input variation, an output with quantifier change is seen more sensitive than the one with only numerical change.

The *joint information gain* $I^{jt}(a_i, a_k)$ is used to quantify the uncertainty reduction by simultaneously knowing two inputs $\mathbf{a}_i$ and $\mathbf{a}_k$ with certainty, and the jth joint information gain is calculated by

$$I_j^{jt}(a_i, a_k) = [\mathfrak{M}(\mathbf{b}_j) - \mathfrak{M}(\mathbf{b}_j \mid a_i, a_k)] \,/\, \mathfrak{M}(\mathbf{b}_j) \quad (22)$$

under the computable conditions $\mathfrak{M}(\mathbf{b}_j) \neq 0$ and $\mathfrak{M}(\mathbf{b}_j) \times \mathfrak{M}(\mathbf{b}_j|a_i, a_k) \geq 0$ where $i = 1, \ldots, n$, $j = 1, \ldots, m$.

The difference between $I_j^{jt}(a_i, a_k)$ and $I_j^m(a_i)$ and $I_j^m(a_k)$ is the extra information gained by the interaction between a_i and a_k. Assuming that a_i and a_k are independent to each other, the interaction between $\mathbf{a}_i$ and $\mathbf{a}_k$, as an indicator of the strength of correlation between the two, is quantified by

$$I_j^{in}(a_i, a_k) = I_j^{jt}(a_i, a_k) - I_j^m(a_i) - I_j^m(a_k) - \alpha\varepsilon \quad (23)$$

where $\alpha\varepsilon$ is a compensation term, and $i \neq k$. The compensation term is introduced because there could be an error when calculating the linear relationship in Equation 23, whereas indeterminacy is defined with the logarithm function. In other words, $\alpha\varepsilon$

is introduced such that $I_j^{in}(a_i, a_k)$ becomes zero when the linear combination of the effects of individual inputs $I_j^m(a_i)$ and $I_j^m(a_k)$ is comparable to $I_j^{jt}(a_i, a_k)$ if there is no interaction between the two inputs. ε is defined as $\varepsilon = [\mathfrak{M}(\mathbf{b}_j|a_i) + \mathfrak{M}(\mathbf{b}_j|a_k) - \mathfrak{M}(\mathbf{b}_j|a_i, a_k) - \mathfrak{M}(\mathbf{b}_j)]/\mathfrak{M}(\mathbf{b}_j)$. The Boolean indicator α is defined as $\alpha = 1$ if $2^{\mathfrak{M}(\mathbf{b}j)} + 2^{\mathfrak{M}(\mathbf{b}j|ai,ak)} = 2^{\mathfrak{M}(\mathbf{b}j|ai)} + 2^{\mathfrak{M}(\mathbf{b}j|ak)}$. Otherwise, $\alpha = 0$.

Equation 23 should also be under the condition that $I_j^{jt}(a_i, a_k)$, $I_j^m(a_i)$ and $I_j^m(a_k)$ are all computable. The *total information gain* $I(a_i)$ with respect to the jth constraint provided by knowing a_i with certainty is defined as

$$I_j(a_i) = I_j^m(a_i) + \sum\nolimits_{k \neq i} I_j^{in}(a_i, a_k) \tag{24}$$

With the above definitions of information gains and quantifier change, we can now perform the sensitivity analysis. There are three possible values of $\mathfrak{M}(\mathbf{b}|a_i)$ as a_i has three choices of representative values. Thus, we use a *sensitivity zone* of $I_j(a_i)$ to represent the possible sensitivities of $\mathbf{a}_i$ with respect to the jth output. The lower and upper bounds of the sensitivity zone are computed from the minimum and maximum among the three values in $\mathrm{I}_j(a_i) = \{I_j(a_i = \inf(\mathbf{a}_i)),\ I_j(a_i = \mathrm{mid}(\mathbf{a}_i)),\ I_j(a_i = \sup(\mathbf{a}_i))\}$. When $I_j(a_i)$ is not computable, the sensitivity zone is computed from the minimum and maximum values in $Q_j(a_i) = \{Q_j(\inf(\mathbf{a}_i)),\ Q_j(\mathrm{mid}(\mathbf{a}_i)),\ Q_j(\sup(\mathbf{a}_i))\}$.

Table 1 lists the rules of sensitivity comparison, where $S_j(a_i)$ denotes sensitivity of the ith input with respect to the jth constraint when $\mathbf{a}_i$ changes to a_i. $\mathrm{mig}(\mathcal{I}_j(a_i)) = \min\{|I_j(\inf(\mathbf{a}_i))|, |I_j(\mathrm{mid}(\mathbf{a}_i))|, |I_j(\sup(\mathbf{a}_i))|\}$. $\mathrm{mag}(\mathcal{I}_j(a_i)) = \max\{|I_j(\inf(\mathbf{a}_i))|, |I_j(\mathrm{mid}(\mathbf{a}_i))|, |I_j(\sup(\mathbf{a}_i))|\}$.

Quantified interval constraints have logic interpretation embedded in the mathematical expression. Therefore, the impact of input variation includes not only the one on the indeterminacy change of the output, but also on the satisfiability of the constraint. The satisfiability of a quantified constraint can be verified by checking if the set intersection (defined in Equation 5) between an initially given output $\mathbf{b}_j^0$ and the computed one $\mathbf{b}_j = \boldsymbol{F}_j(\mathbf{a}_{\sim i}, a_i, \mathbf{x})$ is empty. If the intersection is not empty, then the interpretation exists and constraint is satisfiable. Otherwise, the constraint is unsatisfiable.

Table 1. Sensitivity comparison rules.

$I_j(a_i)$	$I_j(a_k)$	Rules
Comp	Comp	$S_j(a_i) \geq S_j(a_k)$, if $\mathrm{mig}(\mathcal{I}_j(a_i)) \geq \mathrm{mag}(\mathcal{I}_j(a_k))$; $S_j(a_i) < S_j(a_k)$, if $\mathrm{mag}(\mathcal{I}_j(a_i)) < \mathrm{mig}(\mathcal{I}_j(a_k))$; Not decided, otherwise.
Comp	Incomp	$S_j(a_i) < S_j(a_k)$.
Incomp	Comp	$S_j(a_i) > S_j(a_k)$.
Incomp	Incomp	$S_j(a_i) \geq S_j(a_k)$, if $\mathrm{mig}(\mathcal{Q}_j(a_i)) \geq \mathrm{mag}(\mathcal{Q}_j(a_k))$; $S_j(a_i) < S_j(a_k)$, if $\mathrm{mag}(\mathcal{Q}_j(a_i)) < \mathrm{mig}(\mathcal{Q}_j(a_k))$; Not decided, otherwise.

Note: Comp – computable; Incomp – not computable.

The indeterminacy of output intersection $\mathfrak{M}(\mathbf{b}_j^{\cap}|a_i)$ can be measured by Equation 17 in which $\mathbf{b}_j^{\cap} = \mathrm{pro}(\boldsymbol{F}_j(\mathbf{a}_{\sim i}, a_i, \mathbf{x})) \cap \mathrm{pro}(\mathbf{b}_j^0)$. It implies that the indeterminacy of the interaction between the jth given output and the one computed by knowing $\mathbf{a}_i$ with certainty. When $\mathfrak{M}(\mathbf{b}_j^{\cap}|a_i) < 0$, the jth quantified interval constraint is unsatisfiable. The relationship between $u(\cdot)$ and $\mathfrak{M}(\cdot)$ is

$$u_j(\mathbf{a}_{-i}, a_i, \mathbf{x}) = \begin{cases} 1, \text{if } \mathfrak{M}(\mathbf{b}_j^{\cap} \mid a_i) < 0 \\ 0, \text{otherwise} \end{cases} \tag{25}$$

which is the unsatisfiability of the jth constraint.

Similarly, we have $u_j(\mathbf{a}, \mathbf{x})$ with variables $\mathbf{a}$ and $\mathbf{x}$ by computing $\mathfrak{M}(\mathbf{b}_j^{\cap})$ where $\mathbf{b}_j^{\cap} = \mathrm{pro}(\boldsymbol{F}_j(\mathbf{a}, \mathbf{x})) \cap \mathrm{pro}(\mathbf{b}_j^0)$. For example, in a problem with three constraints, we have $u(\mathbf{a}, \mathbf{x}) = (1, 0, 1)$. It indicates that the first and third constraints are not satisfiable. Now with a variation of input $\mathbf{a}_i$, $u(\mathbf{a}_{\sim i}, a_i, \mathbf{x}) = (1, 1, 1)$. We will know that the second constraint becomes unsatisfiable too. The comparison is described as

$$\Delta u_j(\mathbf{a}_i \mapsto a_i) = u_j(\mathbf{a}_{\sim i}, a_i, \mathbf{x}) - u_j(\mathbf{a}, \mathbf{x}) \tag{26}$$

which indicates the unsatisfiability change of the jth constraint from 0 to 1 with the variation $\mathbf{a}_i \mapsto a_i$.

The total unsatisfiability change with a variation of $\mathbf{a}_i \mapsto a_i$, denoted by $\Delta U(\mathbf{a}_i \mapsto a_i)$, is computed as

$$\Delta U(\mathbf{a}_i \mapsto a_i) = \sum_{j=1}^{m} \Delta u_j(\mathbf{a}_i \mapsto a_i) \tag{27}$$

where $\mathbf{a}_i \mapsto a_i$ introduces unsatisfiability to the problem when $\Delta U(\mathbf{a}_i \mapsto a_i) > 0$. Otherwise, it does not affect the unsatisfiability of the problem if $\Delta U(\mathbf{a}_i \mapsto a_i) = 0$. An input with $\Delta U(\cdot) > 0$ has more impact on the output than the one with $\Delta U(\cdot) = 0$. Because a_i can be either one of the three representative values, a set of three values $\Delta\mathcal{U}(\cdot)$ will be obtained, similar to $\mathcal{I}_j(\cdot)$ and $\mathcal{Q}_j(\cdot)$.

The sensitivity analysis includes two aspects. One is based on $I(\cdot)$ that specifies the total information gain of inputs quantitatively. The other is based on $\Delta U(\cdot)$ that provides the unsatisfiability change of the problem qualitatively. The values in $\Delta\mathcal{U}(\cdot)$ are used if the sensitivity levels cannot be decided based on the rules in Table 1. In this case, if $\mathrm{mig}(\Delta\mathcal{U}(\mathbf{a}_i \mapsto a_i)) \geq \mathrm{mag}(\Delta\mathcal{U}(\mathbf{a}_k \mapsto a_k))$, $S(a_i) \geq S(a_k)$. If $\mathrm{mag}(\Delta\mathcal{U}(\mathbf{a}_i \mapsto a_i)) \leq \mathrm{mig}(\Delta\mathcal{U}(\mathbf{a}_k \mapsto a_k))$, $S(a_i) \leq S(a_k)$. Otherwise, the sensitivity is not comparable.

4 NUMERICAL EXAMPLES AND RESULTS

The proposed method for interval-valued quantified constraints is applied to the stability analysis of a soil slope problem (Smith, 2006) and the reliability analysis of a deep excavation (Xu et al., 2011). Here the problems have been modified to estimate the sensitivities of inputs in the form of QCSP.

4.1 *SA for soil slope stability*

In the stability analysis for a soil slope shown in Figure 1, input parameter uncertainties represented by intervals are taken into the calculation of the safety factor F by the Bishop's conventional method as

$$F = \frac{\sum(cl + W(\cos\alpha - r_u \sec\alpha)\tan\phi)}{\sum W \sin\alpha} \tag{28}$$

where W is slice weight, α is the angle between the total normal force acting on base of slice and the vertical, l is the straight distance between the endpoints of slice circle, $r_u = rh_w/zr$ is the pore water pressure to the weight of the material acting on unit area, c is unit cohesion, and ϕ is the angle of shearing resistance. The data are given in Table 2, corresponding to the five slices in Figure 1.

The sensitivities of inputs α, c, r and ϕ are estimated. For each selected input, the value is varying within the range $\pm10\%$ of the initially given value. The interval-valued constraint in Equation 28 can be solved as a quantified one which has a solution set with

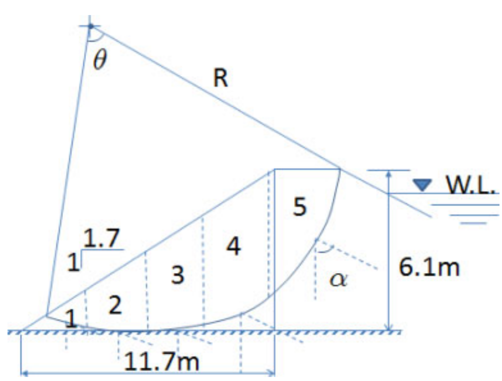


Figure 1. Cross-section of an earth dam.

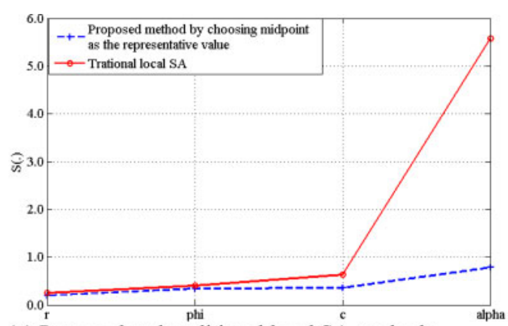
Table 2. Data used in Example 1 (Smith, 2006).

slice	z (m)	b (m)	W (kN)	α (°)	h_w (m)
1	0.95	2.35	42.9	0.985	0.654
2	2.44	2.35	110.1	0.998	1.958
3	3.32	2.35	149.8	0.940	2.440
4	3.50	2.35	157.9	0.819	2.020
5	1.74	2.35	78.5	0.545	0.246
others	c (kPa)	ϕ (°)	r (kN/m^3)	R (m)	θ (°)
	12	20	19.2	9.15	89

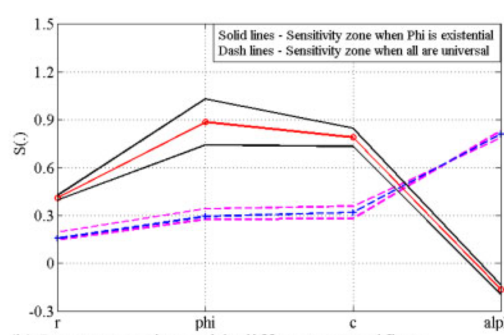
the interpretation $(\forall X_\rho \in \mathbf{X}_\rho^\Delta)(\forall A_\rho \in \mathbf{A}_\rho^\Delta)\ (\exists B_\rho \in \mathbf{B}_\rho^\Delta)$ $f(A_\rho, X_\rho) = B_\rho$, with the input, unknown, and output as $A_\rho = (\alpha, c, r, \phi), X\rho = \theta$, and $B_\rho = F$ respectively.

The result of the proposed method calculated by choosing midpoint as the representative value is compared with the result from the tradition local SA method based on the first derivatives, as shown in Figure 2-a. The quantitative sensitivity zones of $S(\cdot)$ for each input with respect to the output are shown in Figure 2-b. The sensitivity level rankings are listed in Table 3. The rankings marked by * cannot be decided because of the overlaps between the sensitivity zones. When all inputs have the universal quantifier in QCSP, the problem is the same as the classical CSP.

The same input has different sensitivities with respect to an output when it is associated with different quantifiers. For example, in Figure 2-b and rows 2–3 in Table 3, sensitivities are quantitatively different between the case when all inputs are associated the universal quantifier and the one when ϕ becomes existentially quantified, even though their rankings do



(a) Proposed and traditional local SA methods



(b) Inputs associate with different quantifiers

Figure 2. Comparison between the results.

Table 3. Sensitivity ranking in Example 1.

	Inputs	Sensitivity ranking highest←			→lowest
Local SA		α	c	ϕ	r
Proposed	All universal	α	ϕ*	c*	r
	ϕ existential	α	ϕ*	c*	r

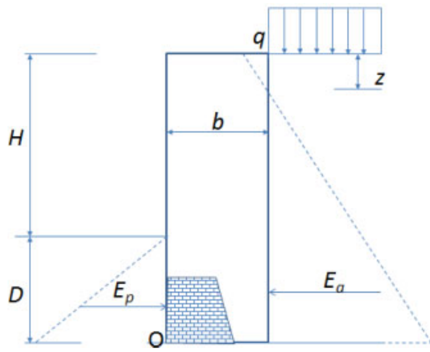


Figure 3. Cross-section of gravity retaining wall.

not change. In Figure 2-b, $S(\alpha)$ is negative. It implies that the change of α to a representative value causes the quantifier change of the output. In this case, $I(\alpha)$ is replaced by $Q(\alpha)$ during the ranking. The change of output quantifiers has more impact than the one with value change only. Therefore, α has the highest sensitivity ranking among all inputs. The change of quantifier for one input may also have impacts on those other inputs which are involved in the same constraint.

4.2 *SA for gravity retaining system of deep excavation*

The proposed method is also applied to the reliability analysis of a deep excavation, as shown in Figure 3. Two performances against sliding and overturning for deep excavation are analyzed. The safety corresponding factors are

$$F_s = (W \tan\varphi + cb + E_p) / E_a - (1+\delta_1)$$

and

$$F_o = \frac{Wb/2 + E_p D/3}{(E_a - K_a qH)(H-z)/3 + K_a qH^2/2} - (1+\delta_2)$$

respectively, where the soil density r, the internal friction angle ϕ, the cohesion c, and the average density of the wall body r_0 are taken as the inputs with uncertainties. H, D, b and q are the design variables. The soil pressures on the retaining structure are calculated by

$$E_a = \frac{1}{2} r(H+D)^2 K_a - 2c(H+D)\sqrt{K_a} + \frac{2c^2}{r} + qK_a(H+D)$$

$$E_p = \frac{1}{2} rD^2 K_a + 2cD\sqrt{K_a}$$

where $K_a = \tan^2(45 - \phi/2)$, $K_p = \tan^2(45 + \phi/2)$, and $z = 2c/(r_0 K_a^{1/2})$. W is the weight of wall. The data used here are given in Table 4. Two scenarios are used to illustrate the proposed method.

Scenario 1: Only one type of quantifiers, either ∀ or ∃, is associated with all elements in **A**, **B** and **X**. The solution set corresponds to the traditional interval constraints without quantifiers.

Table 4. Data used in Example 2 (Xu et al., 2011).

variables	value	variables	value
r (kN/m^3)	[17.8, 18.7]	H (m)	9.8
ϕ (°)	[10.8, 11.7]	D (m)	8.2
c (kPa)	[8.2, 9.1]	b (m)	6.5
r_0 (kN/m^3)	[18, 19]	q (kPa)	10

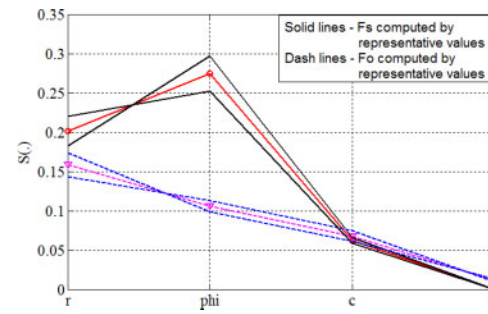


Figure 4. Sensitivities zones of inputs inScenario 1.

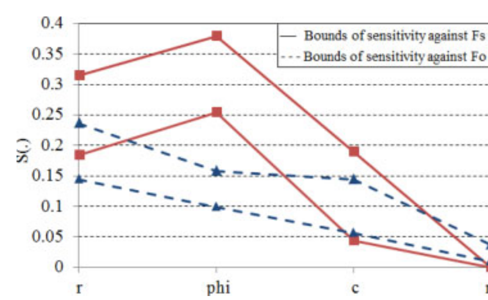


Figure 5. Sensitivity bounds when c is existential in Scenario 2.

Scenario 2: Only one type of quantifiers, either ∀ or ∃, is associated with **B** and **X** whereas both types of quantifiers are associated with different elements in **A**. The multiple occurrence input, c, is chosen to be the one which changes the quantifier from ∀ to ∃ in an assumed design scenario.

Since the existentially quantified variables in the constraint have the meaning that there exists at least one value within the interval range that satisfies the constraints, multiple occurrences of the same existential variable should be avoided, which is usually done by applying the dual operation to all occurrences except one. Therefore, it is possible to have different results when different occurrences are chosen. Here, c appears four times in constraint F_s, and three times in constraint F_o. There are a total of four cases to treat the multiple occurrences of c. The upper and lower bounds of sensitivities of inputs with respect to F_s and F_o are shown in Figure 5.

In Figures 4–5, $S(r_0) = 0$ with respect to the stability against sliding implies that r_0 has no impact on F_s, because it is not involved in constraint F_s. $S(r_0)$ with respect to the stability against overturning is the

Table 5. Sensitivity ranking in Example 2.

Inputs Quantifiers	Outputs	Sensitivity ranking highest←			→lowest
All universal	F_s	ϕ	r	c	r_0
	F_o	r	ϕ	c	r_0
c existential	F_s	r^*	ϕ^*	c	r_0
	F_o	r^*	ϕ^*	c^*	r_0

smallest one among other inputs. r_0 only appears once in F_o whereas other inputs have multiple occurrences. $S(c)$ changed when c is associated with the existential quantifier. The indeterminacy of the whole problem is changed.

In Table 5, when different occurrence of c is chosen to be existential, the four cases of rankings between r and ϕ with respect to F_s are not always the same and cannot be decided. They are marked by * in Table 5. The sensitivity zones of r, ϕ and c overlap with each other so that their rankings cannot be decided either, which are marked by * in the row 4 of Table 5. The sensitivity ranking is not unique for multiple occurrences of the input. Lower and upper bounds can improve the robustness of the analysis.

5 CONCLUSIONS

In this paper, a new global sensitivity analysis method was developed for interval-valued quantified constrains without assuming probabilistic distributions for the input. The effects of both quantifiers and interval values on the constraints were analyzed qualitatively and quantitatively. By the proposed metrics of information gain and unsatisfiability, we can assess the effect of input variations on generalized interval constraints. This approach provides an efficient alternative to the classical variance-based statistical sensitivity analysis.

ACKNOWLEDGMENTS

This research is supported in part by China Scholarship Council with a scholarship (No. 2011613059) to the first author.

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