Semantic Tolerancing with Generalized Intervals

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ABSTRACT

A new tolerance modeling scheme, semantic tolerance modeling, was recently developed to enable interpretable tolerance analysis. In this paper, a new dimensioning and tolerancing practice, semantic tolerancing, is proposed with the theoretical support of semantic tolerance models. Following principles of interpretability, this new tolerancing approach captures more design intent, including flexible material selection, component sorting in selective assembly, rigidity of constraints, and assembly sequence.

Keywords: Semantic tolerance modeling, design intent, modal interval analysis.

1. INTRODUCTION

Tolerance modeling forms an important link between design and manufacturing. A significant amount of research has been done to explore the mathematical basis for dimensional and geometric tolerance representation, analysis, and synthesis. Relationships among tolerances in components and assemblies are formulated in different ways and solved numerically. The typical analysis methods include variational estimation, kinematic formulation, statistical approximation, and Monte Carlo simulation. However, current tolerance modeling methods do not represent the semantics of tolerance specifications well.

First, the traditional tolerance analysis methods assume objects all have rigid geometry. Variance is increasingly stacked up as components are assembled. Geometric variation of assembly is always assumed to be larger than those of its subassemblies and components. This rigid body tolerance analysis overestimates the variations of flexible materials, such as assemblies containing sheet metal and plastic parts.

Second, current modeling and analysis methods do not maintain the semantics of tolerance specifications during model formulation and numerical computing. These specifications, along with the relations among them, imply certain manufacturing and assembly methods, especially the sequence of fabrication as an important component of design intent. Tolerance analysis is usually simplified to computation of numerical intervals. However, the logical dependency and algebraic relations among variations are left out in existing approaches. This leads to the problem that numerical solutions are not interpretable. Instead of focusing only on mathematical and numerical convenience, a good model of tolerance should convey the full semantics of size and geometric tolerances and support analysis and synthesis with a simple yet comprehensive structure.

A semantic tolerance modeling scheme [1, 2] was recently proposed in order to better capture tolerancing intent. It is to embed logical relationships and engineering implications into mathematical representation based on modal interval analysis [3]. Numerical results are interpretable based on several interpretability principles. In this paper, we introduce a new dimensioning and tolerancing practice, semantic tolerancing, based on the semantic tolerance models. It allows us to capture more design intent, such as flexible material selection, component sorting in selective assembly, and assembly sequence. In the remainder of the paper, Section 2 gives a brief overview of existing tolerance models, modal interval analysis, and semantic tolerance modeling. Section 3 describes the basic principles of interpretability in semantic tolerance modeling. Section 4 presents the new semantic tolerancing approach for design specification.

2. BACKGROUND

2.1 Tolerance Modeling

There is plenty of literature on tolerance modeling [4, 5]. In the variational approaches, tolerance zones are established in either 3D Euclidean space or configuration space, such as offsetting tolerance zone [6, 7], plane boundary

representation [8, 9] and volume-based representation [10]. In the statistical approaches [11, 12], geometric and size tolerances are not modeled separately. Linear tolerance stack-up can be estimated using a root-sum-square method whereas non-linear stack-up is approximated using Taylor series. While the root-sum-square method gives optimistic estimation, alternatives were proposed to do adjustment and correction for shifts and drifts [13]. Tolerance zone can be represented in a mean-variance (μ-σ^2) space [14, 15]. In the kinematic approaches, geometrical variation and displacement are modeled by unified vectors and matrices [16, 17], small displacement torsor [18, 19], homogenous matrices [20, 21], kinematic links and adjustment in Euclidean space [22, 23]) and configuration space [24]. In the Monte Carlo simulation approaches [25, 26], the assumption of variable independence is not needed. Based on tolerance response relations, large numbers of samples are randomly generated and evaluated statistically. The drawback is that the computational cost for the sampling process is very high if an accurate estimation is required. The process also depends on the pre-assumption of certain statistical distributions for input random variates. For tolerance analysis of flexible materials, a combination of finite element structural analysis and Monte Carlo simulation has been proposed [27, 28, 29]. These tolerance modeling and analysis methods have been widely accepted and used in commercial software such as Vis VSA, CE/Tol, and CATIA-TAA. However, one critical element is missing. The engineering semantics of the input and output variations during computational analysis is not maintained, and the relations among variations in components and assemblies are not interpretable. The new semantic tolerance modeling is to overcome the deficiency, which is based on generalized intervals.

2.2 Modal Interval Analysis vs. Interval Analysis

Interval mathematics [30] is a generalization in which interval numbers replace real numbers, interval arithmetic replaces real arithmetic, and interval analysis replaces real analysis. The set of real intervals is

$$\mathbb{IR} = \{[a,b] \mid a \in \mathbb{R}, b \in \mathbb{R}\} \tag{1}$$

Let $[a] = [\underline{a}, \overline{a}]$, $[b] = [\underline{b}, \overline{b}]$ be real intervals and $\circ$ be one of the four basic arithmetic operations for real numbers, $\circ \in \{+,-,\cdot,/\}$. The corresponding operations for interval $[a]$ and $[b]$ are defined by

$$[a] \circ [b] = \{x \circ y \mid x \in [a], y \in [b]\} \tag{2}$$

Not only intervals solve the problem of representation for real numbers on a digital scale, but they are the most suitable way to represent uncertainties and errors in technical constructions, measuring, computations, and ranges of fluctuation and variation. Interval analysis has been extensively used in reliable computing in computer science. In engineering fields, methods of interval analysis have been used in robust geometry construction and evaluation [31, 32], set-based modeling [33], imprecise structural analysis [34], design optimization [35], finite-element formulation and analysis [36, 37], soft constraint solving [38, 39], and worst-case tolerance analysis and synthesis [40].

Modal interval analysis (MIA) [3, 41, 42, 43, 44] is a logical and semantic extension of the classical interval analysis (IA). Unlike classical IA which identifies an interval by a set of real numbers as in Eqn. (1), MIA identifies an interval by a set of predicates which is fulfilled by the real numbers. The modal quantifier associates to every real predicate. A real function $f(x)$ where $x \in \mathbb{R}^n$ can be extended to $f(\mathbf{x})$ where $\mathbf{x} \in \mathbb{KR}^n$, which is called $\mathbb{KR}$-extension or AE-extension. In special cases of the real arithmetic operations, i.e., $f(x,y) = x \circ y$ with $\circ \in \{+,-,\times,\div\}$, the $\mathbb{KR}$-extensions lead to the so-called Kaucher arithmetic [45].

Given a set of closed intervals of real numbers in $\mathbb{R}$, the set of logical existential ($\exists$) and universal ($\forall$) quantifiers, each *modal interval* or *generalized interval* $\mathbf{x} := [\underline{x}, \overline{x}] \in \mathbb{KR}$ has an associated quantifier. $\mathbf{x}$ is called *proper* interval when $\underline{x} \le \overline{x}$ and called *improper* interval when $\underline{x} \ge \overline{x}$. The set of proper intervals is denoted by $\mathbb{IR} = \{[\underline{x}, \overline{x}] \mid \underline{x} \le \overline{x}\}$, and the set of improper interval is $\overline{\mathbb{IR}} = \{[\underline{x}, \overline{x}] \mid \underline{x} \ge \overline{x}\}$.

Three special operators, *pro*, *imp*, and *dual*, are defined in Kaucher arithmetic. Given a generalized interval $\mathbf{x} = [\underline{x}, \overline{x}] \in \mathbb{KR}$, $\text{pro}\,\mathbf{x} := [\min(\underline{x}, \overline{x}), \max(\underline{x}, \overline{x})]$ and $\text{imp}\,\mathbf{x} := [\max(\underline{x}, \overline{x}), \min(\underline{x}, \overline{x})]$ return the proper and improper interval values of $\mathbf{x}$ respectively. $\text{dual}[\underline{x}, \overline{x}] := [\overline{x}, \underline{x}]$ builds a relation between proper and improper intervals. Related to arithmetic operations, $(\text{dual}\,\mathbf{x}) \circ (\text{dual}\,\mathbf{y}) = \text{dual}(\mathbf{x} \circ \mathbf{y})$. The *inclusion* relation between modal intervals is defined as $[\underline{x}, \overline{x}] \subseteq [\underline{y}, \overline{y}] \Leftrightarrow \underline{y} \le \underline{x} \wedge \overline{x} \le \overline{y}$. The *less or equal* relation is defined as $[\underline{x}, \overline{x}] \le [\underline{y}, \overline{y}] \Leftrightarrow \underline{x} \le \underline{y} \wedge \overline{x} \le \overline{y}$. Tab. 1 lists the major differences between MIA and IA.

	Classical Interval Analysis	Modal Interval Analysis
Validity	$[3,2]$ is an invalid or empty interval	Both $[2,3]$ and $[3,2]$ are valid intervals
Semantics richness	$[2,3]+[2,4]=[4,7]$ is the only valid relation for $+$, and it only means "stack-up" and "worst-case". $-,\times,/$ are similar.	$[2,3]+[2,4]=[4,7]$, $[2,3]+[4,2]=[6,5]$, $[3,2]+[2,4]=[5,6]$, $[3,2]+[4,2]=[7,4]$ are all valid, and each has a different meaning. $-,\times,/$ have similar semantic properties.
Completeness of arithmetic	$\mathbf{a}+\mathbf{x}=\mathbf{b}$, but $\mathbf{x}\neq\mathbf{b}-\mathbf{a}$ $[2,3]+[2,4]=[4,7]$, $[2,4]\neq[4,7]-[2,3]$ $\mathbf{a}\times\mathbf{x}=\mathbf{b}$, but $\mathbf{x}\neq\mathbf{b}/\mathbf{a}$ $[2,3]\times[3,4]=[6,12]$, $[3,4]\neq[6,12]/[2,3]$ $\mathbf{x}-\mathbf{x}\neq 0$ $[2,3]-[2,3]=[-1,1]\neq 0$	$\mathbf{a}+\mathbf{x}=\mathbf{b}$, and $\mathbf{x}=\mathbf{b}-\mathrm{dual}\,\mathbf{a}$ $[2,3]+[2,4]=[4,7]$, $[2,4]=[4,7]-[3,2]$ $\mathbf{a}\times\mathbf{x}=\mathbf{b}$, and $\mathbf{x}=\mathbf{b}/\mathrm{dual}\,\mathbf{a}$ $[2,3]\times[3,4]=[6,12]$, $[3,4]=[6,12]/[3,2]$ $\mathbf{x}-\mathrm{dual}\,\mathbf{x}=0$ $[2,3]-[3,2]=0$

Tab. 1: The major differences between MIA and traditional IA.

2.3 Semantic Tolerance Modeling
MIA is able to model problems on a logic basis and to obtain the interval functional evaluations for the mathematical model involved. Based on generalized intervals, we proposed a new semantic tolerance modeling scheme, in which the implications of tolerance stacking can be embedded in tolerance models.

The purpose of semantic tolerance modeling is to embed engineering implications into mathematical relations, which is to build a bridge between mathematical theory and engineering practice. Semantic tolerance modeling has two important characteristics: (1) Interpretability: being able to interpret tolerance intervals during the analysis and synthesis processes and to provide the basic understanding of tolerancing semantics; and (2) Optimality: being able to analyze tolerance propagation and accumulation so that tolerances can be specified without losing the basic requirements of completeness and soundness. Interpretability allows tolerance semantics to be embedded in interval results. Optimality assures the tightness of variation estimation.

3. INTERPRETABILITY
The uniqueness of generalized intervals is the semantic extension of intervals with logic quantifiers. As a set of predicates, semantics of an interval $\mathbf{x}\in\mathbb{KR}$ is denoted by $\left(Q_{\mathbf{x}}x\in\mathrm{pro}\,\mathbf{x}\right)$ where $Q_{\mathbf{x}}\in\{\exists,\forall\}$. An interval $\mathbf{x}\in\mathbb{KR}$ is called *existential* if $Q_{\mathbf{x}}=\exists$. Otherwise, it is called *universal* if $Q_{\mathbf{x}}=\forall$. If a real relation $z=f(x_1,\cdots,x_n)$ is extended to the interval relation $\mathbf{z}=\mathbf{f}(\mathbf{x}_1,\cdots,\mathbf{x}_n)$, the interval relation $\mathbf{z}$ is interpretable if there is a semantic relation

$$\left(Q_1\,x_1\in\mathrm{pro}\,\mathbf{x}_1\right)\ldots\left(Q_n\,x_n\in\mathrm{pro}\,\mathbf{x}_n\right)\left(Q_{\mathbf{z}}\,z\in\mathbf{z}\right)\left(z=f(x_1,\cdots,x_n)\right) \tag{3}$$

Two interval extensions of a real function $f(x):\mathbb{R}^n\to\mathbb{R}$, so-called *semantic interval functions*, are defined in a min-max form as the basis of interpretation. They are

$$f^{*}(\mathbf{x}):=[\min_{x_p\in\mathrm{pro}\,\mathbf{x}_p}\max_{x_i\in\mathrm{pro}\,\mathbf{x}_i}f(x_p,x_i),\max_{x_p\in\mathrm{pro}\,\mathbf{x}_p}\min_{x_i\in\mathrm{pro}\,\mathbf{x}_i}f(x_p,x_i)] \tag{4}$$

$$f^{**}(\mathbf{x}):=[\max_{x_i\in\mathrm{pro}\,\mathbf{x}_i}\min_{x_p\in\mathrm{pro}\,\mathbf{x}_p}f(x_p,x_i),\min_{x_i\in\mathrm{pro}\,\mathbf{x}_i}\max_{x_p\in\mathrm{pro}\,\mathbf{x}_p}f(x_p,x_i)] \tag{5}$$

where (x_p,x_i) is the component splitting corresponding to an interval vector $\mathbf{x}=(\mathbf{x}_p,\mathbf{x}_i)$, with the sub-vectors $\mathbf{x}_p$ and $\mathbf{x}_i$ containing the proper and improper components respectively. Important properties of interpretability are available and proved based on these two semantic interval functions.

Theorem 3.1 [3] Given a continuous function $f(\mathbf{x}):\mathbb{R}^n\to\mathbb{R}$ and a generalized interval vector $\mathbf{x}\in\mathbb{KR}^n$, if there exists an interval $\mathbf{f}(\mathbf{x})\in\mathbb{KR}$, then

$$f^{*}(\mathbf{x}) \subseteq \mathbf{f}(\mathbf{x}) \Leftrightarrow \left(\forall x_{p} \in \operatorname{pro}\mathbf{x}_{p}\right)\!\left(Q_{\mathbf{f}}\, z \in \operatorname{pro}\mathbf{f}(\mathbf{x})\right)\!\left(\exists x_{i} \in \operatorname{pro}\mathbf{x}_{i}\right)\!\left(z = f(x_{p}, x_{i})\right) \tag{6}$$

Theorem 3.2 [3] Given a continuous function $f(\mathbf{x}): \mathbb{R}^{n} \to \mathbb{R}$ and a generalized interval vector $\mathbf{x} \in \mathbb{KR}^{n}$, if there exists an interval $\mathbf{f}(\mathbf{x}) \in \mathbb{KR}$, then

$$f^{**}(\mathbf{x}) \supseteq \mathbf{f}(\mathbf{x}) \Leftrightarrow \left(\forall x_{i} \in \operatorname{pro}\mathbf{x}_{i}\right)\!\left(Q_{\operatorname{dual}\mathbf{f}}\, z \in \operatorname{pro}\mathbf{f}(\mathbf{x})\right)\!\left(\exists x_{p} \in \operatorname{pro}\mathbf{x}_{p}\right)\!\left(z = f(x_{p}, x_{i})\right) \tag{7}$$

Let $f : \mathbb{R}^{n} \to \mathbb{R}$ be a rational continuous function. Its modal rational extension $\mathbf{f} : \mathbb{KR}^{n} \to \mathbb{KR}$ replaces the real variables of f with modal interval variables and real operators with interval operators, as originally defined in [45]. The semantics of a modal interval relation or function is embodied in the relation's syntax. The syntax of a function $f(x_{1}, \dots, x_{n}) : \mathbb{R}^{n} \to \mathbb{R}$ can be represented by a syntax tree. For example, the syntax tree of $f_{1} = \left|x_{1} + x_{2}\right| / \left(x_{1} - x_{2}\sqrt{x_{3}}\right)$ is shown in Fig. 1. A component x_{i} is uni-incident in the function $f(x_{1}, \dots, x_{n})$ if it occupies only one leaf of the syntax tree, such as x_{3} in f_{1}. Otherwise, it is multi-incident, such as x_{1} and x_{2} in f_{1}. Leaves and branches of the syntax tree are connected with either one-variable operators such as $\left|\;\right|$ and $\sqrt{\;}$, or two-variable operators such as $+, -, \times, /$.
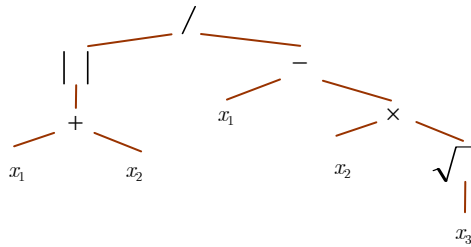

Fig. 1: The syntax tree of $f_{1} = \left|x_{1} + x_{2}\right| / \left(x_{1} - x_{2}\sqrt{x_{3}}\right)$.

3.1 Uni-Incident Interpretation

Theorem 3.3 [3] For a modal rational function $\mathbf{f}(\mathbf{x}) : \mathbb{KR}^{n} \to \mathbb{KR}$, if all arguments of $\mathbf{f}(\mathbf{x})$ are uni-incident, then

$f^{*}(\mathbf{x}) \subseteq \mathbf{f}(\mathbf{x}) \subseteq f^{**}(\mathbf{x})$.

From Theorems 3.1, 3.2, and 3.3, we know modal rational functions of uni-incident variables are interpretable. For example, $f(x, y) = x + y$ is considered for $x \in [1,3]$ and $y \in [2,5]$.

$$\mathbf{f}([1,3],[2,5]) = [1,3] + [2,5] = [3,8],$$
$$\mathbf{f}([1,3],[5,2]) = [1,3] + [5,2] = [6,5],$$
$$\mathbf{f}([3,1],[2,5]) = [3,1] + [2,5] = [5,6],$$
$$\mathbf{f}([3,1],[5,2]) = [3,1] + [5,2] = [8,3],$$

have the meanings of

$$\left(\forall x \in [1,3]\right)\!\left(\forall y \in [2,5]\right)\!\left(\exists z \in [3,8]\right)\!\left(z = x + y\right),$$
$$\left(\forall x \in [1,3]\right)\!\left(\forall z \in [5,6]\right)\!\left(\exists y \in [2,5]\right)\!\left(z = x + y\right),$$
$$\left(\forall y \in [2,5]\right)\!\left(\exists x \in [1,3]\right)\!\left(\exists z \in [5,6]\right)\!\left(z = x + y\right),$$
$$\left(\forall z \in [3,8]\right)\!\left(\exists x \in [1,3]\right)\!\left(\exists y \in [2,5]\right)\!\left(z = x + y\right),$$

respectively.

Different semantics of linear tolerance stack-up in assembly enclosures needs to be differentiated in tolerance design. This includes the semantics associated with assembly sequence, accuracy of tolerance estimation, and controllability of variation.

As tolerances are stacked up in a tolerance chain, the earlier a part is assembled in the sequential process, the less controllable the corresponding variations are in order to close the chain. In an assembly line, correction or adjustment

of individual components in a previously finished subassembly, especially those directly from suppliers, to meet functional requirement is more difficult than the adjustment of newly added ones. In this sense, the tolerances of the earlier assembled parts are out of control of the current worker. They are uncontrollable tolerances. In contrast, the most recently assembled ones have controllable tolerances.

If a dimension is functionally critical and therefore explicitly specified in design and blueprint, it is called working dimension. On the other hand, a dimension is called balance or reference dimension if it is not explicitly specified and its nominal value and tolerance are calculated from working dimensions. Compared to working dimensions, which are hard requirements imposed a priori, balance dimensions are soft and derived a posteriori. In general, *a priori tolerances* are tolerances with predetermined variations, whereas *a posteriori tolerances* are tolerances with derived variations. Whenever defining a relation among tolerances, we have implicitly differentiated these two types of tolerances.

Based on manufacturing and assembly sequences, tolerances may be specified in different ways to designate some desirable semantics. For example, in Fig. 2, dimensions a, b, and c in three components have the relation $a + b = c$. If Part A and B are provided by suppliers and Part C is to be built in house (Fig. 2-a, Case I), or if a and b are working dimensions and c is a balance dimension, the tolerance of c is determined by the tolerances of a and b, and the tolerance chain should be closed. In this case, the semantics of "given A and B, C needs to fit A and B" is expressed as $(\forall a \in \mathrm{pro}\,\mathbf{a})(\forall b \in \mathrm{pro}\,\mathbf{b})(\exists c \in \mathrm{pro}\,\mathbf{c})(a + b = c)$, which is different from the semantics of "given A, B and C need to fit A" when a is a working dimension while b and c are balance dimensions (Fig. 2-b, Case II).

In semantic tolerance models, a priori and a posteriori tolerances are differentiated by the modalities of generalized intervals. A priori tolerances have the semantics of uncontrollable, unchangeable, critical, hard-constrained, specified, etc. For example, tolerances in working dimensions are categorized as a priori. A posteriori tolerances have the semantics of flexible, soft-constrained, adjustable, controllable, feedback, etc. A posteriori variations provide "buffers" in tolerance allocation to make algebraic relations valid and close the tolerance chain, such as balance dimensions. It should be noticed that the semantic categories of a priori and a posteriori tolerances depend on the context of discourse.

Case I: given Part A and Part B, Part C needs to fit A and B.

$(\forall a \in \mathrm{pro}\mathbf{a})(\forall b \in \mathrm{pro}\mathbf{b})(\exists c \in \mathrm{pro}\mathbf{c})(a + b = c)$

(a)

Case II: given Part A, Part B and Part C need to fit A.

$(\forall a \in \mathrm{pro}\mathbf{a})(\exists b \in \mathrm{pro}\mathbf{b})(\exists c \in \mathrm{pro}\mathbf{c})(a + b = c)$

(b)

Case III: given Part C, Part A and Part B need to fit C.

$(\forall c \in \mathrm{pro}\mathbf{c})(\exists a \in \mathrm{pro}\mathbf{a})(\exists b \in \mathrm{pro}\mathbf{b})(a + b = c)$
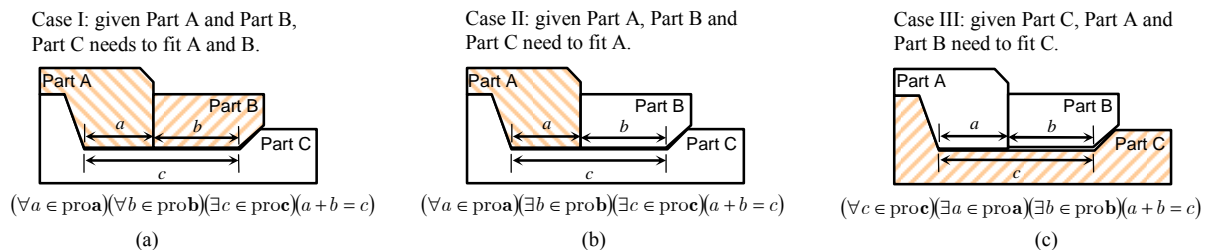
(c)

Fig. 2: Different types of semantics need to be captured, which are not differentiated in traditional modeling methods.

With the symbolic differentiation of a priori and a posteriori tolerances, different strategies of tolerance allocation could be applied in different scenarios. For example, in Fig. 2-a, given two uncontrollable dimensions a and b, the controllable dimension $c = a + b = [2,5] + [1,3] = [3,8]$. In Fig. 2-b, one extra controllable dimension b allows a tighter tolerance of c. $c = a + b = [2,5] + [3,1] = [5,6]$. The tolerance range of c is reduced from 5 to 1, which is smaller than the tolerance range of a. This indicates that the principle of selective assembly may be applied to achieve assembly. Selective assembly is a widely used process of sorting and selecting mating components in pairs so that high-precision assemblies can be achieved even with the low-precision components. This method is valuable when individual components cannot be produced with their tolerances small enough to be fully interchangeable in an assembly, such as some specialized roller bearings with the micrometer level tolerances. However, selective assembly is a manual process, which means it may only be used in low-volume high-value products. In a cost-conscious mass production environment, choosing flexible materials is the alternative, as discussed in Section 3.3.

3.2 Multi-Incident Interpretation

Theorem 3.4 [3] For a modal rational function $\mathbf{f}(\mathbf{x}) : \mathbb{KR}^n \to \mathbb{KR}$, if there are multi-incident improper arguments in $\mathbf{f}(\mathbf{x})$, and $\mathbf{t}^*(\mathbf{x})$ is obtained from $\mathbf{x}$ by transforming, for every multi-incident improper component, all incidences but one into its dual, then $f^*(\mathbf{x}) \subseteq \mathbf{f}(\mathbf{t}^*(\mathbf{x}))$.

Theorem 3.5 [3] For a modal rational function $\mathbf{f}(\mathbf{x}) : \mathbb{KR}^n \to \mathbb{KR}$, if there are multi-incident proper arguments in $\mathbf{f}(\mathbf{x})$, and $\mathbf{t}^{**}(\mathbf{x})$ is obtained from $\mathbf{x}$ by transforming, for every multi-incident proper component, all incidences but one into its dual, then $f^{**}(\mathbf{x}) \supseteq \mathbf{f}(\mathbf{t}^{**}(\mathbf{x}))$.

From Theorems 3.1, 3.2, 3.4, and 3.5, modal rational functions of multi-incident variables are interpretable with some modifications. For example, $f(x,y) = xy/(x+y)$ is extended to $\mathbf{x} = [-1,3]$ and $\mathbf{y} = [15,7]$.

$$\mathbf{f}(\mathbf{x},\mathbf{y}) = [-1,3] \times [15,7]/([-1,3] + [15,7]) = [-0.5,1.5]$$

is not interpretable, whereas

$$\mathbf{f}(\mathbf{t}^*(\mathbf{x},\mathbf{y})) = [-1,3] \times [15,7]/([-1,3] + [7,15]) = [-1.16667,3.5],$$

$$\mathbf{f}(\mathbf{t}^*(\mathbf{x},\mathbf{y})) = [-1,3] \times [7,15]/([-1,3] + [15,7]) = [-1.07143,3.21429],$$

$$\mathbf{f}(\mathbf{t}^{**}(\mathbf{x},\mathbf{y})) = [-1,3] \times [15,7]/([3,-1] + [15,7]) = [-0.388889,1.16667],$$

$$\mathbf{f}(\mathbf{t}^{**}(\mathbf{x},\mathbf{y})) = [3,-1] \times [15,7]/([-1,3] + [15,7]) = [4.5,-1.5]$$

are interpretable. They are interpreted as

$$\left(\forall x \in [-1,3]\right)\left(\exists y \in [7,15]\right)\left(\exists z \in [-1.16667,3.5]\right)\left(z = xy/(x+y)\right),$$

$$\left(\forall x \in [-1,3]\right)\left(\exists y \in [7,15]\right)\left(\exists z \in [-1.07143,3.21429]\right)\left(z = xy/(x+y)\right),$$

$$\left(\forall y \in [7,15]\right)\left(\forall z \in [-0.388889,1.16667]\right)\left(\exists x \in [-1,3]\right)\left(z = xy/(x+y)\right),$$

$$\left(\forall y \in [7,15]\right)\left(\exists x \in [-1,3]\right)\left(\exists z \in [-1.5,4.5]\right)\left(z = xy/(x+y)\right)$$

respectively.

In complex assemblies, parametric relations with multi-incident variables are common. Compared to the traditional tolerance modeling, semantic tolerance modeling allows us to explicitly interpret algebraic relations among variations. Different numerical values and modalities can also be selected in order to derive some specific semantics.

3.3 Rigidity Interpretability

In the material property domain, the variation ranges for rigid materials are corresponding to proper intervals, and the ranges for flexible materials are to improper intervals.

In the one-way clutch example of Fig. 3, the distance vector b, the length of the spring s, and the radius of the ball r have the relation $r + s = b$. If variation ranges $[5.2,5.7]$ and $[7.8,8.0]$ are given to r and b respectively, the range for spring length s can be $[2.1,2.8]$, as in the relation

$$\mathbf{r} + \mathbf{s} = [5.2,5.7] + [2.8,2.1] = [8.0,7.8] = \mathbf{b}.$$

It is interpreted as

$$\left(\forall r \in [5.2,5.7]\right)\left(\forall b \in [7.8,8.0]\right)\left(\exists s \in [2.1,2.8]\right)\left(r + s = b\right).$$

The spring provides a "cushion" to absorb variance. If a larger range $[7.8,8.5]$ is allowed for b, no flexible material is absolutely required for s to absorb the variance, since the relation

$$\mathbf{r} + \mathbf{s} = [5.2,5.7] + [2.6,2.8] = [7.8,8.5] = \mathbf{b}$$

is interpreted as

$$\left(\forall r \in [5.2,5.7]\right)\left(\forall s \in [2.6,2.8]\right)\left(\exists b \in [7.8,8.5]\right)\left(r + s = b\right).$$
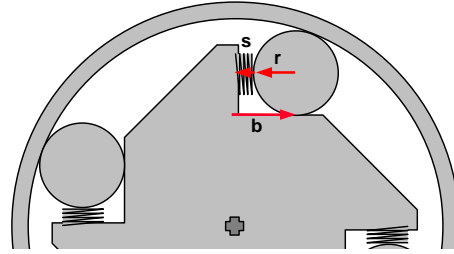
Fig. 3: Variations form a closed loop in assembly.

As illustrated in Fig. 4, the semantic difference between rigid and flexible materials is differentiated by the interval modalities. If the width of the interval $\mathbf{x} = [\underline{x}, \overline{x}]$ is defined as $\mathrm{wid}\,\mathbf{x} := |\overline{x} - \underline{x}|$, the flexibility of materials is quantified by the width of improper intervals. The relative width of an improper interval indicates how flexible the material is. Compressibility may be indicated by the index $\mathrm{I}^-(\mathbf{x}) = (\underline{x} - \overline{x})/\underline{x}$ and stretchability by $\mathrm{I}^+(\mathbf{x}) = (\underline{x} - \overline{x})/\overline{x}$. For example, in Fig. 4-b, the material $\mathbf{x}_1$ is more flexible than the material $\mathbf{x}_2$, and $\mathbf{x}_2$ is more flexible than $\mathbf{x}_3$. The selection of rigid or flexible materials thus can be integrated into algebraic relations.

(a) the improper interval domain corresponds to flexible materials

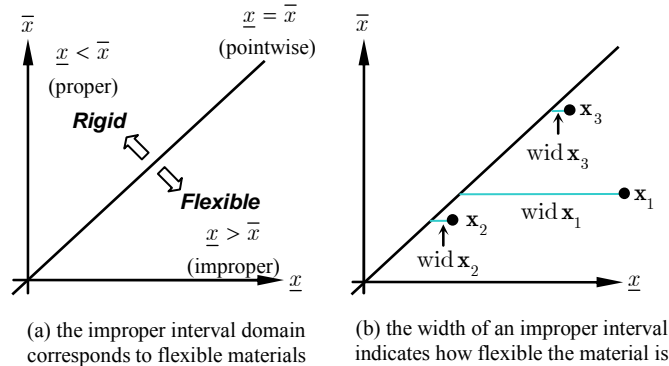(b) the width of an improper interval indicates how flexible the material is

Fig. 4: An inf-sup diagram is also a rigidity diagram.

4. SEMANTIC TOLERANCING

In semantic tolerance models, the algebraic relations among tolerances should be compatible with the semantics of engineering specifications. Based on the interpretability foundation of semantic tolerance modeling, a new dimensioning and tolerancing practice, semantic tolerancing, is proposed. The main difference between the semantic tolerancing and the commonly used tolerancing practice is that a priori and a posteriori tolerances are differentiated in the former one.

The major step of the semantic tolerancing practice is to differentiate a posteriori tolerances from a priori tolerances in symbols. Tolerances with universal modalities are a priori tolerances, and those with existential modalities are a posteriori tolerances. We use a minus-plus notation $x \mp \Delta$ to represent an improper interval and the traditional plus-minus notation $x \pm \Delta$ for proper intervals. If a tolerance value is included by a parenthesis, it is a posteriori. Otherwise it is a priori. If there is a closed tolerance chain $\sum_i d_i = z$ formed, dimensions on the left-hand side with the notation of $d_i \pm \Delta_i$ are a priori tolerances. Those with the notation of $(d_i \mp \Delta_i)$ are a posteriori. On the right-hand side of the chain, notations $(z \pm \Delta)$ and $z \mp \Delta$ are considered to be a posteriori and a priori tolerances respectively. If there is no closed tolerance chain in the drawing, $x \pm \Delta$ denotes a priori tolerance and $(x \mp \Delta)$ denotes a posteriori tolerance. For example, in the drawing of Fig. 5, the tolerance a is a priori, and b is a posteriori. There is a closed tolerance chain $x + y = z$. Therefore, y and z are a priori, and x is a posteriori. In this case, dimensions a, y and z are working

dimensions. b and x are balance dimensions. The closed-loop algebraic relations between working and reference dimensions can now be specified explicitly in drawings.
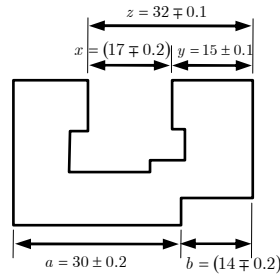

Fig. 5: A priori and a posteriori tolerances in semantic tolerancing.

Assembly sequences can also be inferred from the semantic tolerance chain stack-up. As illustrated in Fig. 6-a, four dimensions a, b, g, and z are specified with a closed chain $a + g + b = z$. Numerically, $[8.8, 9.2] + [2.8, 3.2] + [5.2, 4.8] = [16.8, 17.2]$. $a = 9 \pm 0.2$ and $b = (5 \mp 0.2)$ imply that the subassembly B is assembled after the subassembly A. If the functional requirement of the working dimension g is not met, B needs to be adjusted. However, if the specifications are $a = (9 \mp 0.2)$ and $b = 5 \pm 0.2$ as in Fig. 6-b, A needs to be adjusted to meet the requirement of g. In Fig. 6-c, $g = (3 \mp 0.2)$ indicates that g is no longer functionally critical while a and b are.
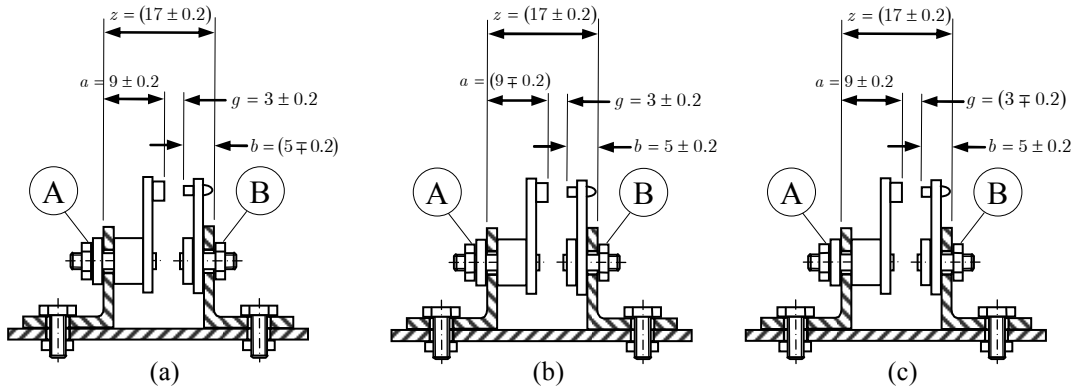

Fig. 6: Semantic tolerancing implies assembly sequence.

In the semantic tolerancing, the flexible and rigid material selection and assembly methods can be explicitly specified. Fig. 7 illustrates the flexible assembly and selective assembly examples of the Case III in Fig. 2-c. The size tolerances of Part A and Part B are a posteriori. Both are larger than the size tolerance of Part C. Yet, three parts need to be assembled. The semantic tolerance symbols in Fig. 7-a indicate that flexible materials with the compressibility index at the level of $(0.3 + 0.3)/10.0 = 0.06$ need to be chosen for Parts A and B. If the variation ranges of a and b are reduced to ∓ 0.03 and the selective assembly process is intended to be used, the a posteriori tolerance symbols capture the intent that A and B need to be sorted and paired, as in Fig. 7-b.

With the differentiation of existential and universal modalities associated with ranges, design intent can be captured in semantic tolerancing, such as how flexible materials are, whether the requirement of a specification is hard or soft, or which sequence to take during the assembly process.

5. CONCLUDING REMARKS
In this paper, a dimensioning and tolerancing scheme, semantic tolerancing, with generalized or modal intervals is proposed. This new tolerancing practice supports the explicit differentiation between a priori and a posteriori tolerances. Based on several interpretability principles, tolerancing semantics can be embedded in algebraic relations in

order to support better design and manufacturing specifications. Symbolically, the new scheme captures more design intent such as physical property difference between rigid and flexible materials, rigidity of requirements, and sequence of assembly.
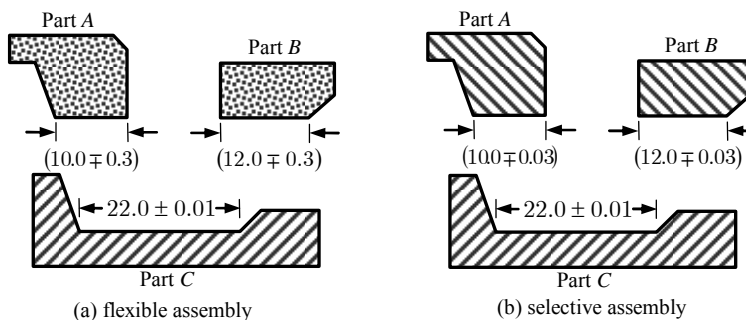

(a) flexible assembly
(b) selective assembly

Fig. 7: Semantic tolerancing captures intent of material selection and selective assembly.

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