Independence in Generalized Interval Probability

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ABSTRACT

Recently we proposed a new form of imprecise probability based on the generalized interval, where the probabilistic calculus structure resembles the traditional one in the precise probability because of the Kaucher arithmetic. In this paper, we study the independence properties of the generalized interval probability. It resembles the stochastic independence with proper and improper intervals and supports logic interpretation. The graphoid properties of the independence are investigated.

INTRODUCTION

Probability theory provides the common ground to quantify uncertainty. However, it has limitations in representing epistemic uncertainty that is due to lack of knowledge. It does not differentiate the total ignorance from other probability distributions, which leads to the Bertrand-style paradoxes such as the Van Fraasen's cube factory (van Fraassen 1989). Probability theory with precise measure also has limitation in capturing indeterminacy and inconsistency. When beliefs from different people are inconsistent, a range of opinions or estimations cannot be represented adequately without assuming some consensus of precise values on the distribution of opinions. Therefore imprecise probabilities have been proposed to quantify aleatory and epistemic uncertainty simultaneously. Instead of a precise value of the probability $P(E)=p$ associated with an event E, a pair of lower and upper probabilities $P(E)=[\underline{p},\overline{p}]$ are used to include a set of probabilities and quantify epistemic uncertainty. The range of the interval $[\underline{p},\overline{p}]$ captures the epistemic uncertainty component and indeterminacy. $P=[0,1]$ accurately represents the total ignorance. When $\underline{p}=\overline{p}$, the degenerated interval probability becomes a precise one. In a general sense, imprecise probability is a generalization of precise probability.

Many representations of imprecise probabilities have been developed. For example, the Dempster-Shafer evidence theory (Dempster 1967; Shafer 1990) characterizes evidence with discrete probability masses associated with a power set of values, where Belief-Plausibility pairs are used to measure uncertainties. The behavioral imprecise probability theory (Walley 1991) models uncertainties with the *lower prevision* (supremum acceptable buying price) and the *upper prevision* (infimum acceptable selling price) following the notations of de Finetti's subjective probability theory. The possibility theory (Dubois and Prade 1988) represents uncertainties with Necessity-Possibility pairs. Probability bound analysis (Ferson et al. 2003) captures uncertain information with pairs of lower and upper distribution functions. F-probability (Weichselberger 2000) represents interval probability as a set of probabilities which maintain the Kolmogorov properties. A random set (Malchanov 2005) is a multi-valued mapping from the probability space to the value space. Fuzzy probability (Möller and Beer 2004) considers probability distributions with fuzzy parameters. A cloud (Neumaier 2004) is a combination of fuzzy sets, intervals, and probability distributions.

Recently we proposed a new form of imprecise probability based on the generalized interval (Wang 2008; 2010), where the probabilistic calculus structure is simplified based on the Kaucher arithmetic (Kaucher 1980). The generalized interval is an extension of the classical set-based interval with enhanced algebraic and semantic properties. Proper and improper interval probabilities are used. In this paper, we study the independence properties of the generalized interval probability.

The concept of independence is essential for the probability theory to decompose a complex problem into simpler and manageable components. Similarly, it is fundamental for imprecise probability theories. Various definitions of independence have been developed, such as epistemic irrelevance and independence (Walley 1991), conformational irrelevance (Levi 1980), mutual independence (Weichselberger 2000), and interval independence (Kuznetsov 1995).

In the remainder of the paper, we first give a brief review of generalized interval. Then the generalized interval probability is introduced. The conditional probability and independence in the generalized interval probability are defined and discussed.

GENERALIZED INTERVAL

In the interval arithmetic, it is guaranteed that the output intervals calculated from the arithmetic include all possible combinations of real values within the respective input intervals. That is, if $[\underline{x},\overline{x}]$ and $[\underline{y},\overline{y}]$ are two real intervals (i.e., $\underline{x},\overline{x},\underline{y},\overline{y} \in \mathbb{R}$) and let $\circ \in \{+,-,\times,/\}$, then we have $\forall x \in [\underline{x},\overline{x}], \forall y \in [\underline{y},\overline{y}], \exists z \in [\underline{x},\overline{x}] \circ [\underline{y},\overline{y}], x \circ y = z$. For example, $[1,3]+[2,4]=[3,7]$ guarantees that $\forall x \in [1,3], \forall y \in [2,4], \exists z \in [3,7], x+y=z$. Similarly, $[3,7]-[1,3]=[0,6]$ guarantees that $\forall x \in [3,7], \forall y \in [1,3], \exists z \in [0,6], x-y=z$. This is an important property that ensures the completeness of range estimations. When input variables are not independent, the output results will over-estimate the actual ranges. This only affects the soundness of estimations, not completeness. Some special techniques also have been developed to avoid over-estimations based on monotonicity properties of functions.

Generalized interval (Gardeñes et al. 2001; Dimitrova et al. 1994) is an extension of the set-based classical interval (Moore 1966) with better algebraic and semantic properties based on the Kaucher arithmetic (Kaucher 1980). A generalized interval $\mathbf{x} := [\underline{x},\overline{x}]$ $(\underline{x},\overline{x} \in \mathbb{R})$ is not constrained by $\underline{x} \le \overline{x}$ any more. Therefore, $[4,2]$ is also a valid interval and called *improper*, while the traditional interval is called *proper*. Based on the Theorems of Interpretability (Gardeñes et al. 2001), generalized interval provides more semantic power to help verify completeness and soundness of range estimations by logic interpretations.

The four examples in Table 1 illustrate the interpretations for operator "+", where the range estimation $[\underline{z},\overline{z}]=[4,7]$ in the 1st row is *complete* and the estimation $[\underline{z},\overline{z}]=[7,4]$ in the 4th row is *sound*. $-,\times,/$ have the similar semantic properties.

Table 1. Illustrations of the semantic extension of generalized interval.

Algebraic Relation: $[\underline{x},\overline{x}]+[\underline{y},\overline{y}]=[\underline{z},\overline{z}]$	Corresponding Logic Interpretation	Quantifier of $[\underline{z},\overline{z}]$	Range Estimation of $[\underline{z},\overline{z}]$
$[2,3]+[2,4]=[4,7]$	$(\forall x\in[2,3])(\forall y\in[2,4])(\exists z\in[4,7])(x+y=z)$	$\exists$	[4,7] *complete*
$[2,3]+\mathbf{[4,2]}=\mathbf{[6,5]}$	$(\forall x\in[2,3])(\forall z\in[5,6])(\exists y\in[2,4])(x+y=z)$	$\forall$	[5,6] *sound*
$\mathbf{[3,2]}+[2,4]=[5,6]$	$(\forall y\in[2,4])(\exists x\in[2,3])(\exists z\in[5,6])(x+y=z)$	$\exists$	[5,6] *complete*
$\mathbf{[3,2]}+\mathbf{[4,2]}=\mathbf{[7,4]}$	$(\forall z\in[4,7])(\exists x\in[2,3])(\exists y\in[2,4])(x+y=z)$	$\forall$	[4,7] *sound*

Compared to the *semi-group* formed by the classical set-based intervals, generalized intervals form a *group*. Therefore, arithmetic operations of generalized intervals are simpler. The set of generalized intervals is denoted by $\mathbb{KR}=\{[\underline{x},\overline{x}]\mid \underline{x},\overline{x}\in\mathbb{R}\}$. The set of proper intervals is $\mathbb{IR}=\{[\underline{x},\overline{x}]\mid \underline{x}\le\overline{x}\}$, and the set of improper interval is $\overline{\mathbb{IR}}=\{[\underline{x},\overline{x}]\mid \underline{x}\ge\overline{x}\}$. The relationship between proper and improper intervals is established with the operator *dual* as $\mathrm{dual}\left[\underline{x},\overline{x}\right]:=\left[\overline{x},\underline{x}\right]$.

The *less than or equal to* partial order relationship between two generalized intervals is defined as

$$\left[\underline{x},\overline{x}\right]\le\left[\underline{y},\overline{y}\right]\Leftrightarrow \underline{x}\le\underline{y}\wedge\overline{x}\le\overline{y} \tag{2.1}$$

The *inclusion* relationship is defined as

$$\left[\underline{x},\overline{x}\right]\subseteq\left[\underline{y},\overline{y}\right]\Leftrightarrow \underline{y}\le\underline{x}\wedge\overline{x}\le\overline{y} \tag{2.2}$$

With the Kaucher arithmetic, generalized intervals form a lattice structure similar to real arithmetic, which is not available in the classical interval arithmetic. This property significantly simplifies the computational requirement. For instance, in classical interval arithmetic, $[0.2,0.3]+[0.2,0.4]=[0.4,0.7]$. However, $[0.4,0.7]-[0.2,0.3]=[0.1,0.5]\ne[0.2,0.4]$. Furthermore, $[0.1,0.2]-[0.1,0.2]=[-0.1,0.1]\ne 0$. In the Kaucher arithmetic, if a *dual* is associated with "–", then $[0.4,0.7]-\mathrm{dual}[0.2,0.3]=[0.4,0.7]-[0.3,0.2]=[0.2,0.4]$. $[0.1,0.2]-\mathrm{dual}[0.1,0.2]=0$. "×" and "÷" are similar.

GENERALIZED INTERVAL PROBABILITY

Definition 1. Given a sample space Ω and a σ-algebra $\mathcal{A}$ of random events over Ω, the generalized interval probability $\mathbf{p}\in\mathbb{KR}$ is defined as $\mathbf{p}:\mathcal{A}\to[0,1]\times[0,1]$ which obeys the axioms of Kolmogorov: (1) $\mathbf{p}(\Omega)=[1,1]$; (2) $[0,0]\le\mathbf{p}(E)\le[1,1]$ $(\forall E\in\mathcal{A})$; and (3) for any countable mutually disjoint events $E_i\cap E_j=\varnothing\ (i\ne j)$, $\mathbf{p}\left(\bigcup_{i=1}^{n}E_i\right)=\sum_{i=1}^{n}\mathbf{p}(E_i)$. Here "≤" is defined as in Eq.(2.1).

Definition 2 (*union*). $\mathbf{p}(A):=\sum_{S\subseteq A}(-\mathrm{dual})^{|A|-|S|}\mathbf{p}(S)$ for $A\subseteq\Omega$.

Definition 3 (*logic coherence constraint*). For a mutually disjoint event partition $\bigcup_{i=1}^{n} E_i = \Omega$, $\sum_{i=1}^{n} \mathbf{p}(E_i) = 1$.

The logic coherent constraint ensures that the imprecise probabilities are logically coherent with precise probabilities. For instance, given that $\mathbf{p}(down) = [0.2, 0.3]$, $\mathbf{p}(idle) = [0.3, 0.5]$, $\mathbf{p}(busy) = [0.5, 0.2]$ for a system's working status, we can interpret it as $(\forall p_1 \in [0.2, 0.3])(\forall p_2 \in [0.3, 0.5])(\exists p_3 \in [0.2, 0.5])(p_1 + p_2 + p_3 = 1)$.

With semantics, we differentiate *non-focal* events ("*busy*" in this example) from *focal* events ("*down*", "*idle*"). An event E is focal if the associated semantics for $\mathbf{p}(E)$ is universal. Otherwise, it is a non-focal if the semantics is existential. While the uncertainties associated with focal events are critical to the analyst, those associated non-focal events are not.

CONDITIONAL PROBABILITY AND CONDITIONAL INDEPENDENCE

The concepts of conditional probability and independence are essential for the classical probability theory. With them, we can decompose a complex problem into simpler and manageable components. Similarly, they are critical for imprecise probabilities. However, there is no agreement on how to define them yet.

Different from all other forms of imprecise probabilities, which are based on convex probability sets, our conditional probability is defined directly from the marginal ones.

Definition 4 (*conditional probability*). $\mathbf{p}(E \mid C) := \mathbf{p}(E \cap C) / \mathrm{dual}\,\mathbf{p}(C)$ $= \left[\underline{p}(E \cap C) / \underline{p}(C), \overline{p}(E \cap C) / \overline{p}(C)\right]$ for all $E, C \in \mathcal{A}$ and $\mathbf{p}(C) > 0$.

Thanks to the algebraic properties of generalized intervals, this definition can greatly simplify computation in applications. In traditional imprecise probabilities, linear and nonlinear programming procedures are heavily dependent upon to compute convex hulls of probability sets. In our definition, only algebraic computation is necessary.

Definition 5. For $A, B, C \in \mathcal{A}$, A is said to be *conditionally independent* with B on C if and only if $\mathbf{p}(A \cap B \mid C) = \mathbf{p}(A \mid C)\mathbf{p}(B \mid C)$.

Definition 6. For $A, B \in \mathcal{A}$, A is said to be *independent* with B if and only if $\mathbf{p}(A \cap B) = \mathbf{p}(A)\mathbf{p}(B)$.

The independence in Definition 5 is a special case of conditional independence in Definition 4, where C is the complete sample space Ω. In addition to computational simplification, our approach also allows for logic interpretation of conditional independence in Definition 4 is interpreted as

$$\left(\forall p_1 \in \mathbf{p}'\left(A \mid C\right)\right)\left(\forall p_2 \in \mathbf{p}'\left(B \mid C\right)\right)\left(\exists p_3 \in \mathbf{p}'\left(A \cap B \mid C\right)\right)\left(p_1 p_2 = p_3\right)$$

This is useful to verify the completeness and soundness of interval bound estimations. The conditional independence in Definition 4 also has a second form, as shown in Theorem 3.1.

Theorem 3.1. For $A, B, C \in \mathcal{A}$, $\mathbf{p}\left(A \cap B \mid C\right) = \mathbf{p}\left(A \mid C\right)\mathbf{p}\left(B \mid C\right) \Leftrightarrow \mathbf{p}\left(A \mid B \cap C\right) = \mathbf{p}\left(A \mid C\right)$.

Proof. $\mathbf{p}\left(A \cap B \mid C\right) = \mathbf{p}\left(A \mid C\right)\mathbf{p}\left(B \mid C\right) \Leftrightarrow$

$\mathbf{p}\left(A \cap B \cap C\right) / \operatorname{dual} \mathbf{p}\left(C\right) = \mathbf{p}\left(A \mid C\right) \cdot \mathbf{p}\left(B \cap C\right) / \operatorname{dual} \mathbf{p}\left(C\right) \Leftrightarrow$

$\mathbf{p}\left(A \cap B \cap C\right) / \operatorname{dual} \mathbf{p}\left(B \cap C\right) = \mathbf{p}\left(A \mid C\right) \Leftrightarrow \mathbf{p}\left(A \mid B \cap C\right) = \mathbf{p}\left(A \mid C\right)$. □

Corollary 3.2 For $A, B, C, D \in \mathcal{A}$ and $A \cap D = \varnothing$, the conditional independence between A and B given C and between A and D given C infers the independence between $A \cup D$ and B given C.

Proof. $\mathbf{p}\left(A \cup D \mid B \cap C\right) = \mathbf{p}\left((A \cup D) \cap B \cap C\right) / \operatorname{dual} \mathbf{p}\left(B \cap C\right)$

$= \left[\mathbf{p}\left(A \cap B \cap C\right) + \mathbf{p}\left(D \cap B \cap C\right)\right] / \operatorname{dual} \mathbf{p}\left(B \cap C\right)$

$= \mathbf{p}\left(A \cap B \cap C\right) / \operatorname{dual} \mathbf{p}\left(B \cap C\right) + \mathbf{p}\left(D \cap B \cap C\right) / \operatorname{dual} \mathbf{p}\left(B \cap C\right)$

$= \mathbf{p}\left(A \mid B \cap C\right) + \mathbf{p}\left(D \mid B \cap C\right) = \mathbf{p}\left(A \mid C\right) + \mathbf{p}\left(D \mid C\right) = \mathbf{p}\left(A \cup D \mid C\right)$ □

The most intuitive meaning of “independence” is that an independence relationship satisfies several *graphoid* properties. With X, Y, Z, W as sets of random variables and “$\perp$” denoting independence, the axioms of graphoid are

(A1) Symmetry: $X \perp Y \mid Z \Rightarrow Y \perp X \mid Z$

(A2) Decomposition: $X \perp \left(W, Y\right) \mid Z \Rightarrow X \perp Y \mid Z$

(A3) Weak union: $X \perp \left(W, Y\right) \mid Z \Rightarrow X \perp W \mid \left(Y, Z\right)$

(A4) Contraction: $\left(X \perp Y \mid Z\right) \wedge \left(X \perp W \mid \left(Y, Z\right)\right) \Rightarrow X \perp \left(W, Y\right) \mid Z$

(A5) Intersection: $\left(X \perp W \mid \left(Y, Z\right)\right) \wedge \left(X \perp Y \mid \left(W, Z\right)\right) \Rightarrow X \perp \left(W, Y\right) \mid Z$

The stochastic independence in precise probability is semi-graphoid satisfying symmetry, decomposition, weak union and contraction. When the probability distributions are strictly positive, intersection is also satisfied. Then, it becomes graphoid. Here, we show that conditional independence in generalized interval probability has these graphoid properties.

Corollary 3.3 (Symmetry) For random variables X, Y, Z , $X \perp Y \mid Z \Rightarrow Y \perp X \mid Z$.

Proof. $X \perp Y \mid Z \Rightarrow \mathbf{p}\left(X = x \cap Y = y \mid Z = z\right) = \mathbf{p}\left(X = x \mid Z = z\right)\mathbf{p}\left(Y = y \mid Z = z\right)$ for any values of $x, y, z \Rightarrow \mathbf{p}\left(Y = y \cap X = x \mid Z = z\right) = \mathbf{p}\left(Y = y \mid Z = z\right)\mathbf{p}\left(X = x \mid Z = z\right) \Rightarrow Y \perp X \mid Z$. □

Remark. If knowing *Y* does not tell us more about *X*, then similarly knowing *X* does not tell us more about *Y*.

Corollary 3.4 (Decomposition) For random variables X,Y,Z,W, $X \perp \left(W,Y\right) | Z \Rightarrow X \perp Y | Z$.

Proof. $X \perp \left(W,Y\right) | Z \quad \Rightarrow \quad \mathbf{p}\left(X = x \cap W = w \cap Y = y \mid Z = z\right) = \mathbf{p}\left(X = x \mid Z = z\right)$ for any values of x, y, z. Since $Y = y$ is equivalent to $\left(W\, has\, all\, possibl\, evalues, Y = y\right)$, $\mathbf{p}\left(X = x \cap Y = y \mid Z = z\right) = \mathbf{p}\left(X = x \cap W = all\, values \cap Y = y \mid Z = z\right) = \mathbf{p}\left(X = x \mid Z = z\right) \quad \Rightarrow$ $X \perp Y | Z$. □

Remark. If combined two pieces of information is irrelevant to X, either individual one is also irrelevant to X.

Corollary 3.5 (Composition) For random variables X,Y,Z,W, $\left(X \perp Y | Z\right) \wedge \left(X \perp W | Z\right) \Rightarrow X \perp \left(W,Y\right) | Z$.

Proof. Because $X \perp Y | Z \Rightarrow$ $\mathbf{p}\left(X = x \cap Y = y \mid Z = z\right) = \mathbf{p}\left(X = x \cap W = all\, values \cap Y = y \mid Z = z\right) = \mathbf{p}\left(X = x \mid Z = z\right)$ and $X \perp W | Z \Rightarrow$ $\mathbf{p}\left(X = x \cap W = w \mid Z = z\right) = \mathbf{p}\left(X = x \cap W = w \cap Y = all\, values \mid Z = z\right) = \mathbf{p}\left(X = x \mid Z = z\right)$, the combination of the above two gives us $\mathbf{p}\left(X = x \cap W = w \cap Y = y \mid Z = z\right) = \mathbf{p}\left(X = x \mid Z = z\right)$, which is $X \perp \left(W,Y\right) | Z$. □

Remark. The combined two pieces of information that are individually irrelevant to X is also irrelevant to X.

Corollary 3.6 (Contraction) For random variables X,Y,Z,W, $\left(X \perp Y | Z\right) \wedge \left(X \perp W | \left(Y,Z\right)\right) \Rightarrow X \perp \left(W,Y\right) | Z$.

Proof. $X \perp W | \left(Y,Z\right)$ and $X \perp Y | Z \quad \Rightarrow \quad \mathbf{p}\left(X \mid W \cap \left(Y \cap Z\right)\right) = \mathbf{p}\left(X \mid Y \cap Z\right) = \mathbf{p}\left(X \mid Z\right)$ $\Rightarrow \quad X \perp \left(W,Y\right) | Z$. □

Remark. If two pieces of information X and Y are irrelevant with prior knowledge of Z and X is also irrelevant to a third piece of information W after knowing Y, then X is irrelevant to both W and Y before knowing Y.

Corollary 3.7 (Reduction) For random variables X,Y,Z,W, $\left(X \perp Y | Z\right) \wedge \left(X \perp \left(W,Y\right) | Z\right) \Rightarrow X \perp W | \left(Y,Z\right)$.

Proof. $X \perp Y | Z$ and $X \perp \left(W,Y\right) | Z \Rightarrow$ $\mathbf{p}\left(X \mid Y \cap Z\right) = \mathbf{p}\left(X \mid Z\right) = \mathbf{p}\left(X \mid \left(W \cap Y\right) \cap Z\right) = \mathbf{p}\left(X \mid W \cap \left(Y \cap Z\right)\right) \Rightarrow X \perp W | \left(Y,Z\right)$. □

Remark. If two pieces of information X and Y are irrelevant with prior knowledge of Z and at the same time X is also irrelevant to both W and Y, then X is irrelevant to the third piece of information W even after knowing Y.

Corollary 3.8 (Weak union) For random variables X, Y, Z, W, $X \perp (W, Y) \mid Z \Rightarrow X \perp W \mid (Y, Z)$

Proof. From the decomposition property in Corollary 3.4, $X \perp (W, Y) \mid Z \Rightarrow X \perp Y \mid Z$. Then from the reduction property in Corollary 3.7, $(X \perp Y \mid Z) \wedge (X \perp (W, Y) \mid Z) \Rightarrow (X \perp W \mid (Y, Z))$. □

Remark. Gaining more information about irrelevant *Y* does not affect the irrelevance between X and W.

Corollary 3.9 (Redundancy) For random variables For random variables X and Y, $X \perp Y \mid X$.

Proof. $\mathbf{p}(Y \mid X \cap X) = \mathbf{p}(Y \mid X) \Rightarrow Y \perp X \mid X \Rightarrow X \perp Y \mid X$ because of symmetry property in Corollary 3.3. □

Corollary 3.10 (Intersection) For random variables X, Y, Z, W, $(X \perp W \mid (Y, Z)) \wedge (X \perp Y \mid (W, Z)) \Rightarrow X \perp (W, Y) \mid Z$.

Proof. $X \perp W \mid (Y, Z) \Rightarrow \mathbf{p}(X \mid W \cap Y = y \cap Z) = \mathbf{p}(X \mid Y = y \cap Z)$ for any y. Therefore, $\mathbf{p}(X \mid W \cap Y = \mathit{all\ values} \cap Z) = \mathbf{p}(X \mid Y = \mathit{all\ values} \cap Z)$. That is, $\mathbf{p}(X \mid W \cap Z) = \mathbf{p}(X \mid Z)$. Then $X \perp Y \mid (W, Z) \Rightarrow \mathbf{p}(X \mid W \cap Y \cap Z) = \mathbf{p}(X \mid W \cap Z) = \mathbf{p}(X \mid Z) \Rightarrow X \perp (W, Y) \mid Z$ □

Remark. If combined information W and Y is relevant to X, then at least either W or Y is relevant to X after learning the other.

Compared to other definitions of independence in imprecise probabilities, the independence defined in generalized interval probability has the most of graphoid properties. Walley's *epistemic irrelevance* (Cozman and Walley 2005) does not have symmetry, whereas the *epistemic independence* as well as Kuznetsov's interval independence (Cozman 2008) do not have the contraction property. Among three possibilistic conditional independence (de Campos and Huete 1999), the two with *not modifying information* comparison operation and with *default conditioning* are not symmetric, whereas the one with *not gaining information* satisfies all.

SUMMARY

In this paper, the conditional independence in a new form of imprecise probability, generalized interval probability, is defined and studied. The generalized interval probability is a generalization of traditional precise probability that considers variability and incertitude simultaneously, in which proper and improper intervals capture epistemic uncertainty. With an algebraic structure similar to the precise probability, generalize interval probability has a simpler calculus structure than other forms of imprecise probabilities. It is shown that the definition of independence in generalized interval probability has graphoid properties similar to the stochastic independence in the precise probability.

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