Imprecise Probabilities with a Generalized Interval Form

Yan Wang

University of Central Florida

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Outline

Interval-based Representations of Imprecise Probabilities

- Dempster-Shafer evidence theory (Dempster, 1967; Shafer, 1976)
- Behavioral imprecise probability theory (Walley, 1991)
- Possibility theory (Zadeh, 1978; Dubois and Prade, 1988)
- Random set (Molchanov, 2005)
- Probability bound analysis (Ferson et al., 2002)
- F-probability (Weichselberger, 2000)
- Fuzzy probability (Möller and Beer, 2004)
- Cloud (Neumaier, 2004)

Engineering Applications of Imprecise Probabilities

- Sensor data fusion (Guede and Girardi, 1997; Elouedi et al., 2004)
- Reliability assessment (Kozine and Filimonov, 2000; Berleant and Zhang, 2004; Coolen, 2004)
- Reliability-based design optimization (Mourelatos and Zhou, 2006; Du et al., 2006)
- Design decision making under uncertainty (Nikolaidis et al., 2004; Aughenbaugh and Paredis, 2006)

Generalized Intervals

- Modal interval analysis (MIA) (Gardenes et al., 2001; Markov, 2001; Shary, 2002; Popova, 2001; Armengol et al., 2001) is an algebraic and semantic extension of interval analysis (IA) (Moore, 1966).
- A modal interval or generalized interval $\mathbf{x} := [\underline{x}, \overline{x}] \in \mathbb{KR}$ is called
  - *proper* when $\underline{x} \leq \overline{x}$. The set of proper intervals is $\mathbb{IR} = \{[\underline{x}, \overline{x}] \mid \underline{x} \leq \overline{x}\}$.
  - *improper* when $\underline{x} \geq \overline{x}$. The set of improper interval is $\overline{\mathbb{IR}} = \{[\underline{x}, \overline{x}] \mid \underline{x} \geq \overline{x}\}$.
  - Operations are defined in Kaucher arithmetic (Kaucher, 1980).
- Not only for outer range estimations, generalized intervals are also convenient for inner range estimations (Kupriyanova, 1995; Kreinovich et al., 1996; Goldsztejn, 2005).

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Generalized/Modal Interval Analysis

- Two operators *pro* and *imp* return proper and improper values respectively, defined as

$$\mathrm{pro}\mathbf{x} := [\min(\underline{x},\overline{x}),\max(\underline{x},\overline{x})] \tag{1}$$

$$\mathrm{imp}\mathbf{x} := [\max(\underline{x},\overline{x}),\min(\underline{x},\overline{x})] \tag{2}$$

- The relationship between proper and improper intervals is established with the operator *dual*:

$$\mathrm{dual}\mathbf{x} := [\overline{x},\underline{x}] \tag{3}$$

- The *inclusion* relation between generalized intervals $\mathbf{x} = [\underline{x},\overline{x}]$ and $\mathbf{y} = [\underline{y},\overline{y}]$ is defined as

$$\begin{aligned} [\underline{x},\overline{x}] \subseteq [\underline{y},\overline{y}] &\iff \underline{x} \geq \underline{y} \wedge \overline{x} \leq \overline{y} \\ [\underline{x},\overline{x}] \supseteq [\underline{y},\overline{y}] &\iff \underline{x} \leq \underline{y} \wedge \overline{x} \geq \overline{y} \end{aligned} \tag{4}$$

- The *less-than-or-equal-to* and *greater-than-or-equal-to* relations are defined as

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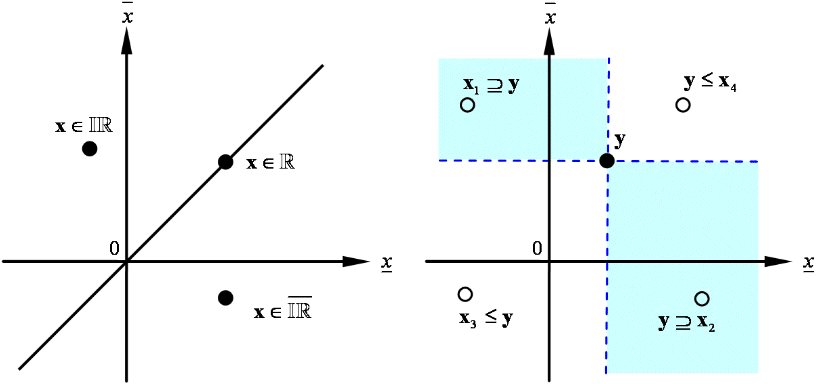
Inf-Sup Diagram



Differences between MIA and Tranditional IA

	Interval Analysis	**Modal Interval Analysis**
Validity	[3,2] is invalid	Both [3,2] and [3,2] are valid intervals
Semantics richness	$[2,3]+[2,4]=[4,7]$ is the only valid relation for $+$, and it only means "stack-up" and worst-case". $-,\times,\div$ are similar.	$[2,3]+[2,4]=[4,7]$, $[2,3]+[4,2]=[6,5]$, $[3,2]+[2,4]=[5,6]$, $[3,2]+[4,2]=[7,4]$ are all valid. The respective meanings are $(\forall a\in[2,3])(\forall b\in[2,4])(\exists c\in[4,7])(a+b=c)$ $(\forall a\in[2,3])(\forall c\in[5,6])(\exists b\in[2,4])(a+b=c)$ $(\forall b\in[2,4])(\exists a\in[2,3])(\exists c\in[5,6])(a+b=c)$ $(\forall c\in[4,7])(\exists a\in[2,3])(\exists b\in[2,4])(a+b=c)$ $.-,\times,\div$ are similar.
Algebraic closure of arithmetic	$\mathbf{a}+\mathbf{x}=\mathbf{b}$, but $\mathbf{x}\neq\mathbf{b}-\mathbf{a}$. $[2,3]+[2,4]=[4,7]$, but $[2,4]\neq[4,7]-[2,3]$ $\mathbf{a}\times\mathbf{x}=\mathbf{b}$, but $\mathbf{x}\neq\mathbf{b}\div\mathbf{a}$. $[2,3]\times[3,4]=[6,12]$, but $[3,4]\neq[6,12]\div[2,3]$ $\mathbf{x}-\mathbf{x}\neq 0$	$\mathbf{a}+\mathbf{x}=\mathbf{b}$, and $\mathbf{x}=\mathbf{b}-\mathrm{dual}\mathbf{a}$. $[2,3]+[2,4]=[4,7]$, and $[2,4]=[4,7]-[3,2]$ $\mathbf{a}\times\mathbf{x}=\mathbf{b}$, and $\mathbf{x}=\mathbf{b}\div\mathrm{dual}\mathbf{a}$. $[2,3]\times[3,4]=[6,12]$, and $[3,4]=[6,12]\div[3,2]$ $\mathbf{x}-\mathrm{dual}\mathbf{x}=0$ $[2,3]-[3,2]=0$

Interval Probability

Definition

- Given a sample space Ω and a σ-algebra $\mathscr{A}$ of random events over Ω,
- The generalized interval probability $\mathbf{p} : \mathscr{A} \mapsto [0,1] \times [0,1]$ obeys the axioms of Kolmogorov:
  - $\mathbf{p}(\Omega) = [1,1] = 1$;
  - $0 \leq \mathbf{p}(E) \leq 1$ $(\forall E \in \mathscr{A})$;
  - For any countable mutually disjoint events $E_i \cap E_j = \emptyset \, (i \neq j)$, $\mathbf{p}(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} \mathbf{p}(E_i)$.

- The lower and upper probabilities here do not have the traditional meanings of lower and upper envelopes:

$$P_*(E) = \inf_{P \in \mathscr{P}} P(E) \qquad P^*(E) = \sup_{P \in \mathscr{P}} P(E)$$

- Rather, they provide the algebraic closure and logical interpretation.

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Operations

Definition

$$\mathbf{p}(E_1 \cup E_2) := \mathbf{p}(E_1) + \mathbf{p}(E_2) - \mathrm{dual}\mathbf{p}(E_1 \cap E_2) \tag{6}$$

- From Eq.(6), we have

$$\mathbf{p}(E_1 \cup E_2) + \mathbf{p}(E_1 \cap E_2) = \mathbf{p}(E_1) + \mathbf{p}(E_2) \tag{7}$$

- Eq.(7) indicates the generalized interval probabilities are 2-monotone (and 2-alternating) in the sense of Choquet's capacities, but stronger than 2-monotonicity.
- Since $\mathbf{p}(E_1 \cap E_2) \geq 0$,

$$\mathbf{p}(E_1 \cup E_2) \leq \mathbf{p}(E_1) + \mathbf{p}(E_2) \tag{8}$$

- The equality of Eq.(8) occurs when $\mathbf{p}(E_1 \cap E_2) = 0$.
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$$\begin{aligned}\mathbf{p}(E_1 \cup E_2 \cup E_3) &= \mathbf{p}(E_1) + \mathbf{p}(E_2) + \mathbf{p}(E_3) - \mathrm{dual}\mathbf{p}(E_1 \cap E_2)\\ &\quad - \mathrm{dual}\mathbf{p}(E_2 \cap E_3) - \mathrm{dual}\mathbf{p}(E_1 \cap E_3) + \mathbf{p}(E_1 \cap E_2 \cap E_3)\end{aligned}$$

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Complement

Definition

$$\mathbf{p}(E^c) := 1 - \mathrm{dual}\mathbf{p}(E) \tag{9}$$

which is equivalent to

$$\mathbf{p}(E) + \mathbf{p}(E^c) = 1 \tag{10}$$

$$\underline{p}(E^c) := 1 - \overline{p}(E) \tag{11}$$

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Logic Coherence Constraint

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- For a mutually disjoint event partition $\bigcup_{i=1}^{n} E_i = \Omega$, we have

$$\sum_{i=1}^{n} \mathbf{p}(E_i) = 1 \tag{13}$$

Suppose $\mathbf{p}(E_i) \in \mathbb{IR}$ (for $i = 1, \ldots, k$) and $\mathbf{p}(E_i) \in \overline{\mathbb{IR}}$ (for $i = k+1, \ldots, n$). Based on the interpretability principles of MIA (Gardenes et al., 2001), Eq.(13) can be interpreted as

$$\begin{aligned}
&\forall p_1 \in \mathbf{p}'(E_1), \ldots, \forall p_k \in \mathbf{p}'(E_k) \\
&\exists p_{k+1} \in \mathbf{p}'(E_{k+1}), \ldots, \exists p_n \in \mathbf{p}'(E_n) \\
&\sum_{i=1}^{n} p_i = 1
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Focal and Non-Focal Events

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- An event E is a *focal* event if its associated semantics is universal ($\mathrm{Q}_{\mathbf{p}(E)} = \forall$).
- Otherwise it is a *non-focal* event if the semantics is existential ($\mathrm{Q}_{\mathbf{p}(E)} = \exists$).

- A *focal* event is an event of *interest*.
- The uncertainties associated with *focal* events are *critical*.
- In contrast, the uncertainties associated with *non-focal* events are "complementary" and "balancing".
- The uncertainties of *non-focal* events are *derived* from those of the corresponding *focal* events.

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Relationships and Interpretations

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- Event E_1 is said to be *less likely* to occur than event E_2, denoted as $E_1 \preceq E_2$, defined as

$$E_1 \preceq E_2 \iff \mathbf{p}(E_1) \leq \mathbf{p}(E_2) \tag{14}$$

- Event E_1 is said to be *less focused* than event E_2, denoted as $E_1 \sqsubseteq E_2$, defined as

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Relationships between a Focal Event and Its Complement

- A focal event E is less likely to occur than its complement if $\mathbf{p}(E) \leq 0.5$; E is more likely to occur than its complement if $\mathbf{p}(E) \geq 0.5$; otherwise, E is more focused than its complement.

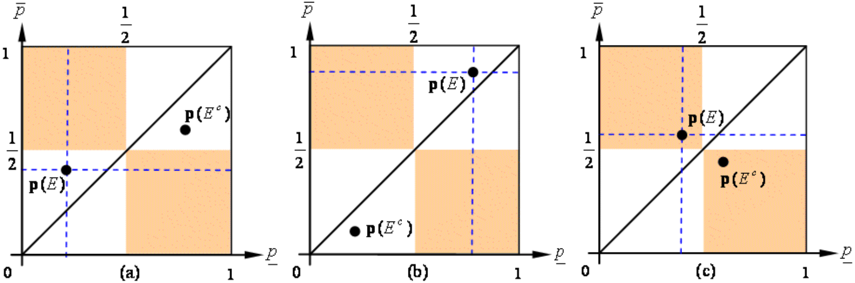

Figure: inf-sup diagrams for different relationships between $\mathbf{p}(E)$ and $\mathbf{p}(E^c)$ when $\mathbf{p}(E) \in \mathbb{IR}$

Conditional Interval Probabilities

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The conditional interval probability $\mathbf{p}(E|C)$ for $\forall E, C \in \mathscr{A}$ is defined as

$$\mathbf{p}(E|C) := \frac{\mathbf{p}(E \cap C)}{\mathrm{dual}\mathbf{p}(C)} = \left[\frac{\underline{p}(E \cap C)}{\underline{p}(C)}, \frac{\overline{p}(E \cap C)}{\overline{p}(C)}\right] \qquad (16)$$

when $\mathbf{p}(C) > 0$.

- The definition is based on marginal probabilities.
- It ensures the *algebraic closure* of the interval probability calculus.
- It is a generalization of the canonical conditional probability in F-probabilities.

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Conditioning Example

Example

$$\mathbf{p}'(E_1) = [0.10, 0.25] \quad \mathbf{p}'(E_2) = [0.20, 0.40] \quad \mathbf{p}'(E_3) = [0.40, 0.60]$$
$$\mathbf{p}'(E_2 \cup E_3) = [0.75, 0.90] \quad \mathbf{p}'(E_1 \cup E_3) = [0.60, 0.80] \quad \mathbf{p}'(E_1 \cup E_2) = [0.40, 0.60]$$

A partition of $\Omega = E_1 \cup E_2 \cup E_3$ is $\mathscr{C} = \{C_1, C_2\}$ where $C_1 = E_1 \cup E_2$ and $C_2 = E_3$.

$$\mathbf{p}(C_1) = [0.40, 0.60], \mathbf{p}(C_2) = [0.60, 0.40]$$

Suppose $\mathbf{p}(E_1) = [0.10, 0.25]$ and $\mathbf{p}(C_1) = [0.60, 0.40]$, we have a *complete* estimation

$$\mathbf{p}(E_1|C_1) = \frac{[0.10, 0.25]}{[0.40, 0.60]} = [0.1666, 0.6250]$$

$$\forall p_{E_1} \in [0.10, 0.25], \forall p_{C_1} \in [0.40, 0.60], \exists p_{E_1|C_1} \in [0.1666, 0.6250], p_{E_1|C_1} = \frac{p_{E_1}}{p_{C_1}}$$

Suppose $\mathbf{p}(E_1) = [0.25, 0.10]$ and $\mathbf{p}(C_1) = [0.40, 0.60]$, we have a *sound* estimation

$$\mathbf{p}(E_1|C_1) = \frac{[0.25, 0.10]}{[0.60, 0.40]} = [0.6250, 0.1666]$$

$$\forall p_{E_1|C_1} \in [0.1666, 0.6250], \exists p_{E_1} \in [0.10, 0.25], \exists p_{C_1} \in [0.40, 0.60], p_{E_1|C_1} = \frac{p_{E_1}}{p_{C_1}}$$

Conditioning Example - cont'd

Example

Suppose $\mathbf{p}(E_1) = [0.25, 0.10]$, $\mathbf{p}(E_2) = [0.20, 0.40]$, and $\mathbf{p}(C_1) = [0.60, 0.40]$, we have

$$\mathbf{p}(E_1|C_1) = \frac{[0.25, 0.10]}{[0.40, 0.60]} = [0.4166, 0.25]$$

$$\mathbf{p}(E_2|C_1) = \frac{[0.20, 0.40]}{[0.40, 0.60]} = [0.3333, 1.0]$$

The interpretations are

$$\forall p_{E_1|C_1} \in [0.25, 0.4166], \forall p_{C_1} \in [0.40, 0.60], \exists p_{E_1} \in [0.10, 0.25], p_{E_1|C_1} = \frac{p_{E_1}}{p_{C_1}}$$

$$\forall p_{E_2} \in [0.20, 0.40], \forall p_{C_1} \in [0.40, 0.60], \exists p_{E_2|C_1} \in [0.3333, 1.0], p_{E_2|C_1} = \frac{p_{E_2}}{p_{C_1}}$$

respectively. Combining the two, we can have the interpretation of

$$\begin{aligned} \forall p_{E_2} \in [0.20, 0.40], \forall p_{C_1} \in [0.40, 0.60], \forall p_{E_1|C_1} \in [0.25, 0.4166], \\ \exists p_{E_1} \in [0.10, 0.25] \exists p_{E_2|C_1} \in [0.3333, 1.0], \\ p_{E_1|C_1} = \frac{p_{E_1}}{p_{C_1}}, p_{E_2|C_1} = \frac{p_{E_2}}{p_{C_1}} \end{aligned}$$

Properties of Conditioning

Independence

If events A and B are independent, then

$$\mathbf{p}(A|B) = \frac{\mathbf{p}(A)\mathbf{p}(B)}{\mathrm{dual}\mathbf{p}(B)} = \mathbf{p}(A) \tag{17}$$

Mutual Exclusion

For a mutually disjoint event partition $\bigcup_{i=1}^{n} E_i = \Omega$, we have

$$\mathbf{p}(A) = \sum_{i=1}^{n} \mathbf{p}(A|E_i)\mathbf{p}(E_i) \tag{18}$$

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Properties of Conditioning

Value of Contradictory Information

If $B \cap C = \emptyset$, $\mathbf{p}(A|C) \subseteq \mathbf{p}(A|B) \Leftrightarrow \mathbf{p}(A|B \cup C) \subseteq \mathbf{p}(A|B)$.

- $\Rightarrow$ If there are two pieces of evidence (B and C), and one (C) may provide a more precise estimation about a focal event (A) than the other (B) may, then the new estimation of probability about the focal event (A) based on the disjunctively combined evidence can be more precise than the one based on only one of them (B), even though the two pieces of information are contradictory to each other.

- $\Leftarrow$ If the precision of the focal event estimation with the newly introduced evidence (C) is improved, the new evidence (C) must be more informative than the old one (B) although these two are controdictory.

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- $\Rightarrow$ If the estimation about a focal event (A) becomes more precise if some new evidence (B) excludes some possibilities (C) from the original evidence ($B \cup C$), then the estimation of probability about the focal event (A) based on the new evidence (B) must be more precise than the one based on the excluded one (C) along.
- $\Leftarrow$ If the precision of the focal event estimation with a contradictory evidence (C) is not improved compared to the old one with another evidence (B), then the new evidence ($B \cup C$) does not improve the estimation of the focal event (A).

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Bayes' Rule with Generalized Intervals

Definition

The Bayes' rule with generalized intervals (GIBR) is defined as

$$\mathbf{p}(E_i|A) = \frac{\mathbf{p}(A|E_i)\mathbf{p}(E_i)}{\sum_{j=1}^{n} \mathrm{dual}\mathbf{p}(A|E_j)\mathrm{dual}\mathbf{p}(E_j)} \tag{19}$$

where $E_i (i = 1, \ldots, n)$ are mutually disjoint event partitions of Ω and $\sum_{j=1}^{n} \mathbf{p}(E_j) = 1$.

$$[\underline{p}(E_i|A), \overline{p}(E_i|A)] = \left[\frac{\underline{p}(A|E_i)\underline{p}(E_i)}{\sum_{j=1}^{n} \underline{p}(A|E_j)\underline{p}(E_j)}, \frac{\overline{p}(A|E_i)\overline{p}(E_i)}{\sum_{j=1}^{n} \overline{p}(A|E_j)\overline{p}(E_j)} \right] \tag{20}$$

- Algebraically consistent with the conditional definition in Eq.(16)

$$\sum_{j=1}^{n} \mathrm{dual}\mathbf{p}(A|E_j)\mathrm{dual}\mathbf{p}(E_j) = \sum_{j=1}^{n} \mathrm{dual}\left[\mathbf{p}(A|E_j)\mathbf{p}(E_j)\right] = \mathrm{dual}\sum_{j=1}^{n} \mathbf{p}(A \cap E_j) = \mathrm{dual}\mathbf{p}(A)$$

2-Monotone Tight Envelope Equivalency

When $n = 2$, $\mathbf{p}(E) + \mathbf{p}(E^c) = 1$. Let $\mathbf{p}(E^c) \in \mathbb{IR}$. Eq.(19) becomes

$$\underline{p}(E|A) = \frac{\underline{p}(A|E)\underline{p}(E)}{\underline{p}(A|E)\underline{p}(E) + \underline{p}(A|E^c)\underline{p}(E^c)} = \frac{\underline{p}(A \cap E)}{\underline{p}(A \cap E) + \underline{p}(A \cap E^c)} \quad (21)$$

$$\overline{p}(E|A) = \frac{\overline{p}(A|E)\overline{p}(E)}{\overline{p}(A|E)\overline{p}(E) + \overline{p}(A|E^c)\overline{p}(E^c)} = \frac{\overline{p}(A \cap E)}{\overline{p}(A \cap E) + \overline{p}(A \cap E^c)} \quad (22)$$

When $\mathbf{p}(A \cap E) \in \mathbb{IR}$ and $\mathbf{p}(A \cap E^c) \in \mathbb{IR}$, the relation is equivalent to the well-known *2-monotone tight envelope* (Fagin and Halpern, 1991; de Campos et al., 1990; Wasserman and Kadan, 1990; Jaffray, 1992; Chrisman, 1995), given as:

$$P_*(E|A) = \frac{P_*(A \cap E)}{P_*(A \cap E) + P^*(A \cap E^c)} \quad (23)$$

$$P^*(E|A) = \frac{P^*(A \cap E)}{P^*(A \cap E) + P_*(A \cap E^c)} \quad (24)$$

where P_* and P^* are the lower and upper probability bounds defined in the traditional interval probabilities.

When $n = 2$, $\mathbf{p}(E) + \mathbf{p}(E^c) = 1$. Let $\mathbf{p}(E^c) \in \overline{\mathbb{IR}}$. Eq.(19) becomes

$$\underline{p}(E|A) = \frac{\underline{p}(A|E)\underline{p}(E)}{\underline{p}(A|E)\underline{p}(E) + \underline{p}(A|E^c)\underline{p}(E^c)} = \frac{\underline{p}(A \cap E)}{\underline{p}(A \cap E) + \underline{p}(A \cap E^c)} \quad (21)$$

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When $\mathbf{p}(A \cap E) \in \mathbb{IR}$ and $\mathbf{p}(A \cap E^c) \in \overline{\mathbb{IR}}$, the relation is equivalent to the well-known *2-monotone tight envelope* (Fagin and Halpern, 1991; de Campos et al., 1990; Wasserman and Kadan, 1990; Jaffray, 1992; Chrisman, 1995), given as:

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where P_* and P^* are the lower and upper probability bounds defined in the traditional interval probabilities.

$$\mathbf{p}(A|E) \subseteq \mathbf{p}(A|E^c) \Leftrightarrow \mathbf{p}(E|A) \subseteq \mathbf{p}(E).$$
$$\mathbf{p}(A|E) \supseteq \mathbf{p}(A|E^c) \Leftrightarrow \mathbf{p}(E|A) \supseteq \mathbf{p}(E).$$

Suppose the likelyhood functions $\mathbf{p}(A|E)$ and $\mathbf{p}(A|E^c)$ as well as prior and posterior probabilities are proper intervals. If the likelyhood estimation of event A given E occurs is more accurate than that of event A given event E does not occur, then the extra information A can reduce the ambiguity of the prior estimation.

$$\mathbf{p}(A|E) \geq \mathbf{p}(A|E^c) \Leftrightarrow \mathbf{p}(E|A) \geq \mathbf{p}(E).$$
$$\mathbf{p}(A|E) \leq \mathbf{p}(A|E^c) \Leftrightarrow \mathbf{p}(E|A) \leq \mathbf{p}(E).$$

If the occurance of event E increases the likelyhood estimation of event A compared to the one without the occurance of event E, then the extra information A will increase the probability of knowing that event E occurs.

Properties of Updating

$$\mathbf{p}(A|E) = \mathbf{p}(A|E^c) \Leftrightarrow \mathbf{p}(E|A) = \mathbf{p}(E).$$

The extra information A does not add much value to the assessment of event E if we have very similar likelyhood ratios, $\mathbf{p}(A|E)$ and $\mathbf{p}(A|E^c)$.

Properties of Updating

Sequence-Independence

$$\mathbf{p}(E|A \cap B) = \mathbf{p}(E \cap B|A)/\mathrm{dual}\mathbf{p}(B|A)$$

$$\mathbf{p}(A \cap B) = \mathbf{p}(B|A)\mathbf{p}(A)$$

The posterior lower and upper bounds obtained by applying a series of evidences sequencially agree with the bounds obtained by conditioning the prior with all of the evidences in a single step.

Soundness of Posterior Probability Estimation

$$\mathbf{p}(E_i|A) = \frac{\mathbf{p}(A|E_i)\mathbf{p}(E_i)}{\sum_{j=1}^{n} \mathrm{dual}\mathbf{p}(A|E_j)\mathrm{dual}\mathbf{p}(E_j)}$$

- Soundness can be verified when $\mathbf{p}(A|E_i) \in \overline{\mathbb{IR}}$, $\mathbf{p}(E_i) \in \overline{\mathbb{IR}}$, $\mathbf{p}(A|E_j) \in \overline{\mathbb{IR}}\,(j = 1, \ldots, n, j \neq i)$, $\mathbf{p}(E_j) \in \overline{\mathbb{IR}}\,(j = 1, \ldots, n, j \neq i)$, and $\mathbf{p}(E_i|A) \in \overline{\mathbb{IR}}$

$$\begin{aligned} \forall_{j \neq i} p_{A|E_j} \in \mathbf{p}'(A|E_j), \forall_{j \neq i} p_{E_j} \in \mathbf{p}'(E_j), \forall p_{E_i|A} \in \mathbf{p}'(E_i|A), \\ \exists p_{A|E_i} \in \mathbf{p}'(A|E_i), \exists p_{E_i} \in \mathbf{p}'(E_i), \\ p_{E_i|A} = \frac{p_{A|E_i} p_{E_i}}{\sum_{j=1}^{n} p_{A|E_j} p_{E_j}} \end{aligned} \tag{25}$$

Summary

- We differentiate focal events from non-focal events by the modalities and semantics of interval probabilities. An event is focal when the semantics associated with its interval probability is universal, whereas it is non-focal when the semantics is existential.
- This differentiation allows us to have a simple and unified representation based on a logic coherence constraint, which is a stronger restriction than the regular 2-monotoniciy.
- Algebraic closure of the new interval form simplifies the calculus.
- It is also shown that the new Bayes' updating rule is a generalization of the 2-monotone tight envelope updating rule under the new representation.
- Logic interpretation helps to verify completeness and soundness of range estimations.