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Semantic Tolerance Modeling based on Modal Interval Analysis

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Abstract: A significant amount of research efforts has been given to explore the mathematical basis for 3D dimensional and geometric tolerance representation, analysis, and synthesis. However, engineering semantics is not maintained in these mathematic models. It is hard to interpret calculated numerical results in a meaningful way. In this paper, a new semantic tolerance modeling scheme based on modal interval is proposed to improve interpretability of tolerance modeling. With logical quantifiers, semantic relations between tolerance specifications and implications of tolerance stacking are embedded in the mathematic model. The model captures the semantics of physical property difference between rigid and flexible materials as well as tolerancing intents such as sequence of specification, measurement, and assembly. Compared to traditional methods, the semantic tolerancing allows us to estimate true variation ranges such that feasible and complete solutions can be obtained.

Keywords: 3D Tolerance Modeling, Engineering Design, Interpretability, Semantic Tolerancing, Interval Analysis, Modal Interval

1 Introduction

Tolerance modeling forms an important link between design and manufacturing processes. A significant amount of research efforts has been given to explore the mathematical basis for 3D dimensional and geometric tolerance representation, analysis, and synthesis. Problems of tolerance relations can be mathematically formulated and solved in different ways. The typical methods for analysis include variational estimation, kinematic formulation, statistical approximation, and Monte Carlo simulation. However, current tolerance modeling methods do not represent the semantics of tolerance specifications well.

First, traditional tolerance analysis methods assume objects have rigid geometry. Variance is increasingly "stack-up" as components are assembled. As shown in *Figure 1*, tolerance of assembly is always assumed to be larger than its subassembly. Rigid body tolerance analysis over-estimates variations of flexible materials, such as assemblies containing sheet metal, polymer, and plastic parts, which are common in aerospace, automobile, and electronics industry. For example, an airplane skin can be slightly warped, and yet it can be riveted in place. Similarly, subassembly components of auto body with much larger variation than the specified can still

achieve the final assembly specification. The conventional addition theorem of variance is no longer valid in these applications. Given the specification of an assembly, unreasonably tight tolerance requirements will be assigned to subassemblies and components during tolerance synthesis, as shown in *Figure 2*. The tolerance allocation based on the rigid body assumption increases manufacturing costs unnecessarily. These methods treat tolerances for rigid and compliant assemblies with the same scheme of +/- range. This does not capture the physical property difference between rigid and flexible materials and implied engineering meanings.

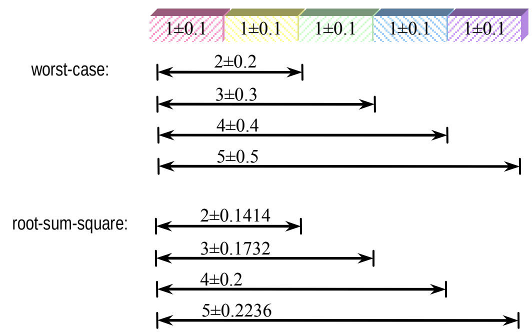


Figure 1. Tolerance ranges are monotonously increasing as assembly is built based on the rigid-body assumption

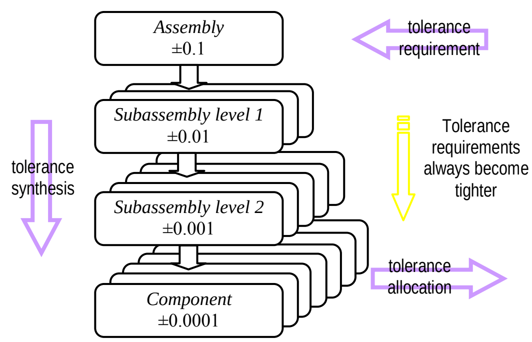


Figure 2. Tolerancing may become so tight that costs increase unnecessarily in flexible assembly based on current rigid-body tolerance synthesis schemes

Second, current tolerance modeling and analysis methods do not maintain the semantics of tolerance specifications. Two types of variation, *priori* and *posteriori*, are not differentiated in current tolerance models. Priori variation is predetermined but unknown, such as tolerances of components from suppliers. On the other hand, posteriori variation is known and controllable, such as tolerances of components built in-house. Engineering implication and tolerance allocation strategies are different for two types of variation. Priori variation is not controllable, while posteriori variation provides “buffers” in tolerance allocation. Further, how to interpret the numerical outputs with high and low bounds is important to understand the relation between tolerances. Numerical results of current methods are not interpretable. Engineering semantics needs to be maintained during mathematical computation.

Third, accuracy of range estimation is essential in tolerance analysis. Basic questions include *completeness* and *feasibility*. Complete solution includes all possible occurrences, which is to check if an interval includes all possible results. Feasible solution does not include impossible occurrences, which is to check if the interval over estimates the range. Current methods except Monte Carlo simulation with extensive sampling do not always give true range. The worst-case method tends to over estimate because of dependency between variables. Statistical methods do not give true range but statistical intervals. The results from vector and kinematic approaches are numerical estimations from algebraic approximations such as linearization. True range estimation should be both complete and feasible.

Instead of focusing only on mathematic and numerical convenience, a good mathematic model of tolerance should convey the full semantics of size and geometric tolerances and support analysis and synthesis with a simple yet comprehensive structure. Existing research does not concentrate on engineering semantics of tolerance zones. This leads to the problem that numerical solutions are not interpretable.

In this paper, we propose a new scheme to represent and analyze tolerance based on modal interval analysis. Extended from traditional set-based interval, modal interval introduces logical quantifiers and provides interpretation of intervals. Tolerancing semantics thus can be integrated into numerical calculation. In addition to better interpretability, modal interval analysis also provides better variation estimation than traditional interval analysis. The remainder of the paper is organized as follows. Section 2 gives an overview of related work on tolerance modeling and interval analysis, and an introduction to modal interval. Section 3 and 4 present the concept of semantic tolerance modeling and its two basic properties: interpretability and optimality. Section 5 describes analysis methods of the semantic tolerance model.

2 Background

2.1 3D Tolerance Modeling

There is plenty of literature on tolerance modeling (Hong and Chang, 2002; ADCATS). We just have a brief overview of 3D geometric tolerance zone representation related to the tolerance semantics. In the variational approaches, tolerance zones are established in 3D Euclidean space by parameter variation of spatial constraints and equations. Requicha (1983) proposed to construct tolerance zones by offsetting the part’s nominal boundaries. Inui *et al*. (1993)

approximate tolerance zone using boundary offset and geometric constraints. Roy and Li (1998; 1999) model tolerance zones of size, flatness, and parallelism in the variational form of plane equation. Teck *et al*. (2001) represent flatness of non-rectangular planar surfaces. Davidson *et al*. (Davidson et al., 2002; Mujezinovic *et al*., 2004) developed a hypothetical volume-based algebraic model to represent size, form, and orientation tolerances. Bhide *et al*. (2003) extended the method for cylindrical features.

In the statistical approaches (Nigam and Turner, 1995; Gerth, 1997), linear tolerance stack-up can be estimated using root-sum-square methods while non-linear stack-up is approximated using Taylor series. Typically it is assumed that the parameters are independent and the random variables are normally distributed. While the root-sum-square gives optimistic estimation, alternatives were proposed to do adjustment and correction for shifts and drifts (Chase and Greenwood, 1988). Srinivasan and O'Connor (1994) model and analyze tolerance based on statistical tolerance zone in the mean-variance (μ-σ^2) space, which is directly related to process capability indices in industry practices. Zhang *et al*. (1999) apply distribution function zone to tolerance synthesis. Different from other approaches, research in the statistical approach concentrates on dimensional tolerance stack-up and geometric tolerances are not modeled separately.

In the kinematic approaches, geometrical variation and displacement are modeled mathematically in vectors and matrices. Vectorial tolerancing (Wirtz *et al*., 1993; Martinsen, 1995) models size, form, location, and orientation tolerances in a unified vector format in order to provide an integrated quality control loop. Small displacement torsor method (Bourdet and Ballot, 1995; Giordano and Duret, 1993) approximates rotation and translation displacement in the form of torsors. Matrix representation method (Whitney *et al*., 1994; Desrochers and Riviere, 1997) models displacement in the form of homogenous transformation matrices. Rivest *et al*. (1994) exploit the kinematic character of the link imposed by a tolerance between the datum and the toleranced feature. Chase *et al*. (Chase *et al*., 1996; Gao *et al*., 1998) perform analysis of assembly using small kinematic adjustment between components based on linear approximation of implicit dimensional constraint functions. Joskowicz *et al*. (Joskowicz et al. 1997; Sack and Joskowicz, 1998) compute contact tolerance zones of planar parametric parts within configuration space. The kinematic methods distinguish size and each type of geometric tolerances. However, relations between variations are not modeled, and estimation result is hard to interpret.

In the Monte Carlo simulation approach (e.g. Turner and Wozny, 1987; Gao *et al*., 1995; Ashiagbor *et al*., 1998), no assumptions on independence and distribution are needed. Based on tolerance response relation, large amount of samples are randomly generated and evaluated in statistical estimation. The drawback is that the computational time for the required sampling process is high if good estimation is needed. It also depends on the pre-assumption of certain statistical distributions for input variables.

The above modeling and analysis methods have been widely accepted and used in commercial software such as Vis VSA® and CE/Tol®. However, it is not easy to interpret the meanings of the specifications for each type of tolerances in component and assembly. Furthermore, the rigid-body assumption tends to over-estimate the variation of flexible materials.

2.2 TOLERANCE ANALYSIS FOR FLEXIBLE ASSEMBLY

There is relatively little research on tolerance analysis for flexible materials. Takezawa (1980) applied linear regression models to predict auto body panel (sheet metal parts) assembly variation using real production data, and he found variation of assembly could be smaller than individual parts. He concluded that "the conventional addition theorem of variance is no longer valid for deformable sheet metal assemblies".

Liu and Hu (1997) proposed a linear finite element structural model to predict variation of sheet metal joining based on the concepts of mechanistic variation simulation and influence coefficient. Monte Carlo simulation is used to randomly displace nodes in a finite element model and the variance of the assembly can be estimated (Liu *et al*., 1996). Long and Hu (1998) extended the method to include the variation of fixtures during assembly operations. Camelio *et al*. (2003) extended the method to multi-station assembly systems with compliant parts. Camelio *et al*. (2004) further applied principle component analysis to simplify covariance matrix in variance computation.

Merkley *et al*. (Merkley *et al*., 1996; Merkley, 1998) developed a finite element tolerance analysis method for flexible assemblies based on linear elastic contact assumption. Polynomial interpolation is used to model geometric covariance between nodes, and stiffness matrix describes material covariance. Bihlmaier (1999) extended the method to consider autocorrelation in geometric covariance matrices.

The above finite element approaches have been integrated into some commercial software such as vis VSA and CATIA-TAA. However, tradeoff between fidelity and performance is always related to finite element methods. The computation becomes very expensive if the variance estimation involves complex assemblies. In most cases, accurate calculation of structural deformation and stress distribution is not the main purpose of tolerance analysis. Confidence of producibility and associated cost analysis need to be estimated without significant computation.

2.3 INTERVAL ANALYSIS

Interval mathematics is a generalization in which interval numbers replace real numbers, interval arithmetic replaces real arithmetic, and interval analysis replaces real analysis. The real number system $\boldsymbol{R}$ is geometrically complete for numerical representation, but not practical for digital computing. Not only intervals solve the problem of representation for real numbers on a digital scale, but they are the most suitable way to represent uncertainties and errors in technical constructions, measuring, computations, and ranges of fluctuation and variation.

The set of intervals corresponding to real numbers is $I(\boldsymbol{R})$. Let $[a]=[\underline{a},\overline{a}]$, $[b]=[\underline{b},\overline{b}]$ be real intervals and $\circ$ be one of the four basic arithmetic operations for real numbers, $\circ \in \{+,-,\cdot,/\}$. The corresponding operations for interval $[a]$ and $[b]$ are defined by

$$[a]\circ[b]=\{x\circ y \mid x\in[a], y\in[b]\}.$$

Interval analysis has been extensively used in reliable computing in computer science. In engineering fields, methods of interval analysis have been used in computer graphics (Mudur and Koparkar, 1984; Toth, 1985; Moore and Wilhelms, 1988; Duff, 1992; Snyder, 1999), robust geometry construction and evaluation (Abrams *et al*., 1998; Shen and Patrikalakis, 1998; Tuohy *et al*., 1997; Wallner *et al*., 2000), set-based modeling (Finch and Ward, 1997), imprecise structural analysis (Rao and Berke, 1997), design optimization (Rao and Cao, 2002), finite-element formulation and analysis (Muhanna and Mullen, 1999; 2001; Muhanna *et al*., 2004), solving soft geometric constraint and preference (Wang, 2004; Wang and Nnaji, 2006), and worst-case tolerance analysis and synthesis (Yang *et al*., 2000).

Interval analysis has intrinsic uncertainty and variance properties for tolerance analysis. However, it is based on a worst-case scenario as in traditional linear stack-up methods. The results usually are pessimistic in this variance addition scheme if dependency exists between variables. In contrast, modal interval analysis is an extension of the traditional interval analysis, which differentiates semantics of interval specification in different application situations.

2.4 Modal Interval Analysis

Modal interval analysis (MIA) (Gardenes *et al*., 2001; Popova, 2001; Armengol *et al*., 2001) is a logical and semantic extension of traditional interval analysis. MIA extends real numbers to intervals. Unlike classical interval analysis which identifies an interval by a set of real numbers, MIA identifies the intervals by the set of predicates which is fulfilled by the real numbers.

Given the set of closed intervals of $\boldsymbol{R}$, $I(\boldsymbol{R})$, and the set of logical existential (E or $\exists$) and universal (U or $\forall$) quantifiers, a modal interval is defined by a pair:

$$X := (X', Q_{\mathrm{X}})$$

in which $X' \in I(\boldsymbol{R})$ and $Q_{\mathrm{X}} \in \{\mathrm{E},\mathrm{U}\}$. X' is the classic interval and Q_X is one of the two modalities.

Similar to the way in which real numbers are associated in pairs with same absolute value but opposite + and − signs, modal intervals are associated in pairs too. Each member of a pair is corresponding to the same closed interval of real line, but having opposite modalities of existential or universal. The quantifiers are operators which transform real predicates into interval predicates. They are written as $\mathrm{E}(x,X')P(x)$ and $\mathrm{U}(x,X')P(x)$, indicating both arguments, the real index x and the interval argument X'. The notations $\mathrm{E}(x,X')$ and $\mathrm{U}(x,X')$ are interpreted as $(\exists x \in X')$ and $(\forall x \in X')$ respectively.

The canonical notation for modal interval is

$$[a,b] := \begin{cases} ([a,b]',\mathrm{E}) & \text{if } a \le b \\ ([b,a]',\mathrm{U}) & \text{if } a \ge b \end{cases}.$$

A modal interval $([a,b]',\mathrm{E})$ is called *existential* or *proper* interval whereas $([b,a]',\mathrm{U})$ is called *universal* or *improper* interval. The set of modal intervals is denoted by $I^*(\boldsymbol{R})$. The modal quantifier Q is associated with every real predicate $P(.)$. For a variable $x \in \boldsymbol{R}$ and $(X',\mathrm{Q_x}) \in I^*(\boldsymbol{R})$, Q is interpreted by $\mathrm{Q_x}$ as

$$\mathrm{Q}(x,(X',\mathrm{Q_x}))P(x) := \mathrm{Q_x}(x,X')P(x).$$

Predicates of modal intervals are defined as the set of real predicates.

$$Pred(X',\mathrm{Q_x}) := \{P(.) \in Pred(\boldsymbol{R}) \mid \mathrm{Q}(x,(X',\mathrm{Q_x}))P(x)\}.$$

Based on the above semantic extension, basic arithmetic operations of modal interval are defined as follows. For A = $[a_1,a_2]$ and B = $[b_1,b_2]$,

$A+B=[a_1+b_1,a_2+b_2]$, $A-B=[a_1-b_2,a_2-b_1]$

$$A\times B = \begin{cases}
[a_1b_1,a_2b_2] & (a_1\ge 0,a_2\ge 0,b_1\ge 0,b_2\ge 0)\\
[a_1b_1,a_1b_2] & (a_1\ge 0,a_2\ge 0,b_1\ge 0,b_2< 0)\\
[a_2b_1,a_2b_2] & (a_1\ge 0,a_2\ge 0,b_1< 0,b_2\ge 0)\\
[a_2b_1,a_1b_2] & (a_1\ge 0,a_2\ge 0,b_1< 0,b_2< 0)\\
[a_1b_1,a_2b_1] & (a_1\ge 0,a_2< 0,b_1\ge 0,b_2\ge 0)\\
[\max(a_1b_1,a_2b_2),\min(a_2b_1,a_1b_2)] & (a_1\ge 0,a_2< 0,b_1\ge 0,b_2< 0)\\
[0,0] & (a_1\ge 0,a_2< 0,b_1< 0,b_2\ge 0)\\
[a_2b_2,a_1b_2] & (a_1\ge 0,a_2< 0,b_1< 0,b_2< 0)\\
[a_1b_2,a_2b_2] & (a_1< 0,a_2\ge 0,b_1\ge 0,b_2\ge 0)\\
[0,0] & (a_1< 0,a_2\ge 0,b_1\ge 0,b_2< 0)\\
[\min(a_1b_2,a_2b_1),\max(a_1b_1,a_2b_2)] & (a_1< 0,a_2\ge 0,b_1< 0,b_2\ge 0)\\
[a_2b_1,a_1b_1] & (a_1< 0,a_2\ge 0,b_1< 0,b_2< 0)\\
[a_1b_2,a_2b_1] & (a_1< 0,a_2< 0,b_1\ge 0,b_2\ge 0)\\
[a_2b_2,a_2b_1] & (a_1< 0,a_2< 0,b_1\ge 0,b_2< 0)\\
[a_1b_2,a_1b_1] & (a_1< 0,a_2< 0,b_1< 0,b_2\ge 0)\\
[a_2b_2,a_1b_1] & (a_1< 0,a_2< 0,b_1< 0,b_2< 0)
\end{cases}
\quad
A/B = \begin{cases}
[a_1/b_2,a_2/b_1] & (a_1\ge 0,a_2\ge 0,b_1>0,b_2>0)\\
[a_2/b_2,a_1/b_1] & (a_1\ge 0,a_2\ge 0,b_1<0,b_2<0)\\
[a_1/b_2,a_2/b_2] & (a_1\ge 0,a_2<0,b_1>0,b_2>0)\\
[a_2/b_1,a_1/b_1] & (a_1\ge 0,a_2<0,b_1<0,b_2<0)\\
[a_1/b_1,a_2/b_1] & (a_1<0,a_2\ge 0,b_1>0,b_2>0)\\
[a_2/b_2,a_1/b_2] & (a_1<0,a_2\ge 0,b_1<0,b_2<0)\\
[a_1/b_1,a_2/b_2] & (a_1<0,a_2<0,b_1>0,b_2>0)\\
[a_2/b_1,a_1/b_2] & (a_1<0,a_2<0,b_1<0,b_2<0)
\end{cases}$$

The inclusion relation between modal intervals is defined as $[a_1,a_2] \subseteq [b_1,b_2] \Leftrightarrow (a_1 \ge b_1, a_2 \le b_2)$. Semantically, $A \subseteq B \Leftrightarrow Pred(A) \subseteq Pred(B)$. If $A \subseteq B$, the implication $\mathrm{Q}(x,A)P(x) \Rightarrow \mathrm{Q}(x,B)P(x)$ is valid. The "less or equal" relation is defined as $[a_1,a_2] \le [b_1,b_2] \Leftrightarrow (a_1 \le b_1, a_2 \le b_2)$. Some modal interval operations are defined as $Prop([a_1,a_2]) := [\min(a_1,a_2),\max(a_1,a_2)]$, $Impr([a_1,a_2]) := [\max(a_1,a_2),\min(a_1,a_2)]$, and $Width([a_1,a_2]) := |a_1 - a_2|$.

MIA is able to model problems on a logical basis and to obtain the interval functional evaluations for the mathematical model involved. Based on modal interval, we propose a new semantic tolerance modeling scheme, in which the implications of tolerance stacking can be embedded in the tolerance model. Accurate range estimation can be achieved compared to traditional worst-case interval methods.

The purpose of semantic tolerance modeling is to capture logical therefore engineering meanings and implications in mathematical representation, which is to build a bridge between mathematic theory and engineering practice. Semantic tolerance modeling has two important characteristics: (1) *Interpretability*: being able to interpret tolerance intervals during analysis and synthesis processes and to provide the basic understanding of tolerancing semantics; and (2) *Optimality*: being able to analyze tolerance propagation and accumulation so that tolerances can be specified without losing the basic requirements of completeness and feasibility. Interpretability allows tolerance semantics to be embedded in interval results. Optimality assures tightness of variation estimation. The following sections will describe the properties of modal interval representation in semantic tolerancing.

3 Interpretability

The uniqueness of modal interval is the modal semantic extension. If a real relation $z = f(x_1, \cdots, x_n)$ is extended to the interval relation $Z = F(f)(X_1, \cdots, X_n)$, the interval relation Z is interpretable if there is a semantic relation

$$\mathrm{Q}_1(x_1, X_1)\cdots\mathrm{Q}_n(x_n, X_n)\mathrm{Q}_z(z, F(f)(X_1, \cdots, X_n))z = f(x_1, \cdots, x_n).$$

A component *x* is *uni-incident* in a function *f*(X) if it occupies only one leaf of the syntax tree for the function. Otherwise, it is *multi-incident*. To reduce the interdependency effect of multi-incidence, which usually over estimates interval function ranges, two interval extensions of real function $f(\mathbf{x})$, so-called semantic interval functions, are defined in min-max situation as:

$$f^*(\mathbf{X}) := [\min_{x_p \in X'_p} \max_{x_i \in X'_i} f(x_p, x_i), \max_{x_p \in X'_p} \min_{x_i \in X'_i} f(x_p, x_i)],$$

$$f^{**}(\mathbf{X}) := [\max_{x_i \in X'_i} \min_{x_p \in X'_p} f(x_p, x_i), \min_{x_i \in X'_i} \max_{x_p \in X'_p} f(x_p, x_i)],$$

where (x_p, x_i) is the component splitting corresponding to interval vector $\mathbf{X} = (X_p, X_i)$, with X_p and X_i are sub-vectors containing proper and improper components respectively.

Important properties of interpretability are available and proved.

Theorem 3.1 (Gardenes *et al*., 2001) Given a continuous function $f : \boldsymbol{R}^n \to \boldsymbol{R}$ and a modal vector $\mathbf{X} \in I^*(\boldsymbol{R}^n)$, if there exists an interval $F(\mathbf{X}) \in I^*(\boldsymbol{R})$, then

$$f^*(\mathbf{X}) \subseteq F(\mathbf{X}) \Leftrightarrow \mathrm{U}(x_p, X'_p)\mathrm{Q}(z, F(\mathbf{X}))\mathrm{E}(x_i, X'_i)z = f(x_p, x_i).$$

Theorem 3.2 (Gardenes *et al*., 2001) Given a continuous function $f:\boldsymbol{R}^n \to \boldsymbol{R}$ and a modal vector $\mathbf{X} \in I^*(\boldsymbol{R}^n)$, if there exists an interval $F(\mathbf{X}) \in I^*(\boldsymbol{R})$, then

$$f^{**}(\mathbf{X}) \supseteq F(\mathbf{X}) \Leftrightarrow \mathrm{U}(x_i, X'_i)\mathrm{Q}(z, Dual(F(\mathbf{X})))\mathrm{E}(x_p, X'_p)z = f(x_p, x_i),$$

where Dual operator is defined as $Dual([a,b]) := [b,a]$.

3.1 UNI-INCIDENT INTERPRETATION

Let $f:\boldsymbol{R}^n \to \boldsymbol{R}$ be a rational continuous function. Its modal rational extension $fR: I^*(\boldsymbol{R}^n) \to I^*(\boldsymbol{R})$ is simply replacing the real variables of *f* with modal interval variables.

Theorem 3.3 (Gardenes *et al*., 2001) For a modal rational function $fR(\mathbf{X})$, if all arguments of $fR(\mathbf{X})$ are uni-incident, then

$$f^*(\mathbf{X}) \subseteq fR(\mathbf{X}) \subseteq f^{**}(\mathbf{X}).$$

From Theorems 3.1, 3.2, and 3.3, we know modal rational functions of uni-incident variables are interpretable. For example, $f(x,y) = x + y$ is considered for X' = [1,3]' and Y' = [2,5]'.

$$\begin{aligned}
&\mathrm{fR}([1,3], [2,5]) = [1,3] + [2,5] = [3,8],\\
&\mathrm{fR}([1,3], [5,2]) = [1,3] + [5,2] = [6,5],\\
&\mathrm{fR}([3,1], [2,5]) = [3,1] + [2,5] = [5,6],\\
&\mathrm{fR}([3,1], [5,2]) = [3,1] + [5,2] = [8,3],
\end{aligned}$$

have the meanings of

$$\mathrm{U}(x,[1,3]')\mathrm{U}(y,[2,5]')\mathrm{E}(z,[3,8]')z = x+y \text{ or } \forall x \in [1,3]', \forall y \in [2,5]', \exists z \in [3,8]', z = x+y,$$

$$\mathrm{U}(x,[1,3]')\mathrm{U}(z,[5,6]')\mathrm{E}(y,[2,5]')z = x+y \text{ or } \forall x \in [1,3]', \forall z \in [5,6]', \exists y \in [2,5]', z = x+y,$$

$$\mathrm{U}(y,[2,5]')\mathrm{E}(x,[1,3]')\mathrm{E}(z,[5,6]')z = x+y \text{ or } \forall y \in [2,5]', \exists x \in [1,3]', \exists z \in [5,6]', z = x+y,$$

$$\mathrm{U}(z,[3,8]')\mathrm{E}(x,[1,3]')\mathrm{E}(y,[2,5]')z = x+y \text{ or } \forall z \in [3,8]', \exists x \in [1,3]', \exists y \in [2,5]', z = x+y,$$

respectively.

Different semantics of linear tolerance stack-up in assembly enclosure needs to be differentiated. For example, in Figure 3, dimensions *a*, *b*, and *c* in three components have relation *a* + *b* = *c*. According to different assembly sequences or manufacturing needs, we may specify tolerances in different ways. If Part A and B are provided by suppliers and Part C is built in house (Figure 3-b, Case I), the tolerance of *c* is determined by the tolerances of *a* and *b*. In this case, the semantics of "given A and B, C needs to fit A and B" is expressed as $\forall a \in A', \forall b \in B', \exists c \in C', a + b = c$, which is different from the semantics of "given A, B and C need to fit A" when Part A is supplied and Part B and C are built in house (Figure 3-c, Case II). The relations between tolerances should be compatible with the semantics of specifications. In the semantic tolerance model, priori and posteriori tolerances are differentiated. In Case I, a and b have priori tolerances, while c has a posteriori tolerance. With the modal extension, the semantics of specification sequence and rational can be embedded in the model.

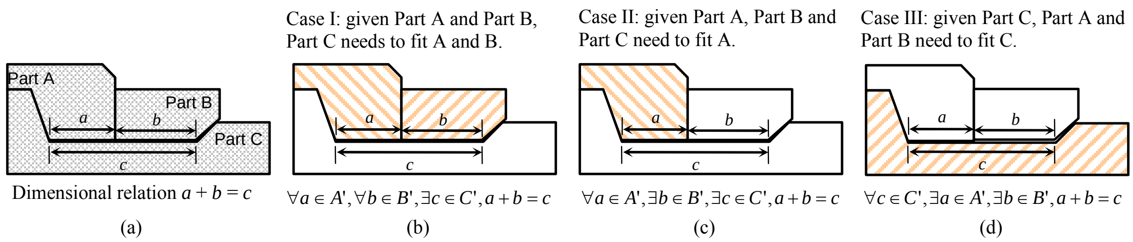


Figure 3. Different types of semantics need to be captured, which are not differentiated in traditional modeling methods

With the differentiation of priori and posteriori tolerances, strategy of tolerance allocation could vary in different scenarios. For example, in Figure 3-b, given two “uncontrollable” dimensions a and b, the “controllable” dimension $c = a + b = [2,5] + [1,3] = [3,8]$. In Figure 3-c, one extra controllable dimension b allows a tighter tolerance of c. $c = a + b = [2,5] + [3,1] = [5,6]$. The tolerance range of c is reduced from 5 to 1, which is smaller than the tolerance range of a. This implies that the principle of selective assembly can be applied to achieve assembly.

3.2 MULTI-INCIDENT INTERPRETATION

Theorem 3.4 (Gardenes *et al.*, 2001) For a modal rational function $fR(\mathbf{X})$, if there are multi-incident improper arguments in $fR(\mathbf{X})$ and $\mathbf{XT}^*$ is obtained from $\mathbf{X}$, by transforming, for every multi-incident improper component, all incidences but one into its dual, then

$$f^*(\mathbf{X}) \subseteq fR(\mathbf{XT}^*).$$

Theorem 3.5 (Gardenes *et al.*, 2001) For a modal rational function $fR(\mathbf{X})$, if there are multi-incident proper arguments in $fR(\mathbf{X})$ and $\mathbf{XT}^{**}$ is obtained from $\mathbf{X}$, by transforming, for every multi-incident proper component, all incidences but one into its dual, then

$$f^{**}(\mathbf{X}) \supseteq fR(\mathbf{XT}^{**}).$$

From Theorems 3.1, 3.2, 3.4, and 3.5, modal rational functions of multi-incident variables are interpretable with some modification. For example, $f(x, y) = xy/(x + y)$ is extended to $X = [-1,3]$ and $Y = [15,7]$.

$$fR(\mathbf{X}) = [-1,3] \times [15,7]/([-1,3] + [15,7]) = [-0.5,1.5]$$

is not interpretable, whereas

$$fR(\mathbf{XT}^{*}) = [-1,3] \times [15,7] / ([-1,3] + [7,15]) = [-1.16667, 3.5],$$
$$fR(\mathbf{XT}^{*}) = [-1,3] \times [7,15] / ([-1,3] + [15,7]) = [-1.07143, 3.21429],$$
$$fR(\mathbf{XT}^{**}) = [-1,3] \times [15,7] / ([3,-1] + [15,7]) = [-0.388889, 1.16667],$$
$$fR(\mathbf{XT}^{**}) = [3,-1] \times [15,7] / ([-1,3] + [15,7]) = [4.5, -1.5]$$

are interpretable. They are interpreted as

$$\mathrm{U}(x,[-1,3]')\mathrm{E}(y,[7,15]')\mathrm{E}(z,[-1.16667,3.5]')z = xy/(x+y),$$
$$\mathrm{U}(x,[-1,3]')\mathrm{E}(y,[7,15]')\mathrm{E}(z,[-1.07143,3.21429]')z = xy/(x+y),$$
$$\mathrm{U}(x,[-1,3]')\mathrm{E}(y,[7,15]')\mathrm{E}(z,[-0.388889,1.16667]')z = xy/(x+y),$$
$$\mathrm{U}(x,[-1,3]')\mathrm{U}(z,[-1.5,4.5]')\mathrm{E}(y,[7,15]')z = xy/(x+y)$$

respectively.

Combining the first three results, we have

$$\mathrm{U}(x,[-1,3]')\mathrm{E}(y,[7,15]')\mathrm{E}(z,[-0.388889,1.16667]')z = xy/(x+y),$$

In assembly, parametric relations with multi-incident variables are common. Compared to traditional tolerance modeling, semantic tolerance modeling allows us to interpret explicit algebraic relations with the interpretability properties of modal intervals. Different numerical values can also be selected in order to derive specific semantics.

3.3 RIGIDITY INTERPRETATION

While existential intervals are looked as “fluctuation” or “autonomous” ranges, universal intervals are regard as “regulating” or “feedback” ranges. In material property domain, tolerance range for rigid material is corresponding to existential interval and flexible material is to universal interval.

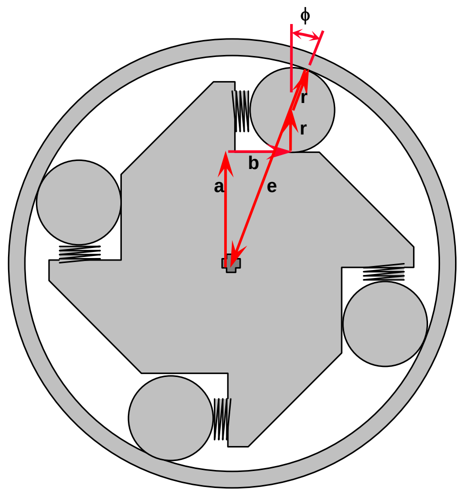


Figure 4. variations of size and geometry, shape deformation, and kinematics form a closed loop in assembly

In the one-way clutch example of Figure 4, the distance vector *b*, the length of the spring *s*, and the radius of the ball *r* satisfy the relation *r*+*s*=*b*. If ranges [5.2,5.7]' and [7.8,8.0]' are given to *r* and *b* respectively, the range for spring length *s* can be [2.1,2.8]', as in relation

$$R + S = [5.2,5.7] + [2.8,2.1] = [8.0,7.8] = B .$$

It is interpreted as

$$\mathrm{U}(r,[5.2,5.7]')\mathrm{U}(b,[7.8,8.0]')\mathrm{E}(s,[2.1,2.8]')r + s = b .$$

The spring provides a "cushion" to absorb variance. If a larger range [7.8,8.5]' is allowed for *b*, no flexible material is required to absorb variance. Rigid material instead of spring for *s* can be chosen, as in relation

$$R + S = [5.2,5.7] + [2.6,2.8] = [7.8,8.5] = B .$$

It is interpreted as

$$\mathrm{U}(r,[5.2,5.7]')\mathrm{U}(s,[2.6,2.8]')\mathrm{E}(b,[7.8,8.5]')r + s = b .$$

As illustrated in Figure 5, the semantic difference between rigid and flexible material is differentiated by interval modality. Selection of rigid or flexible materials is integrated into algebraic relation.

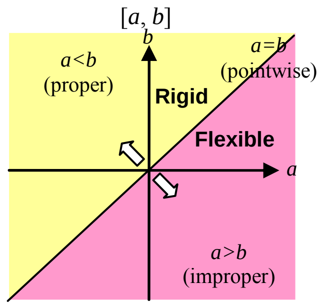


Figure 5. Rigidity diagram

3.4 Semantic Tolerancing

With modal extension, engineering semantics such as sequences of specification, manufacturing, and assembly, as well as material properties can be captured. Tolerance semantics can be grouped into existential and universal categories, including tolerancing intent, specification precedence and dependency, as well as differentiation of constraint and preference. Taxonomy of specification semantics thus can be developed. Some examples of such semantics are listed in *Table 1*. Semantic pairs exist in the domains of supply management, manufacturing and assembly sequences, etc.

Table 1. Examples of tolerance semantics

Domain	*Existential* or *Proper* category	*Universal* or *Improper* category
Supply management	Pre-determined, Uncontrollable, Supplied	Un-determined, Controllable, Built
Manufacturing sequence	Working dimension, Clearance	Balance dimension, Stock removal
Assembly sequence	Place, Virtual condition size	Fit, Bonus tolerance
Material property	Rigid, Wearable	Flexible, Deformable
Process control	Open loop, Manual mode	Closed loop, Auto mode

4 Optimality

For every $\mathbf{X} \in I^*(\boldsymbol{R}^n)$, if the modal rational extension $fR(\mathbf{X})$ satisfy $f^*(\mathbf{X}) = fR(\mathbf{X}) = f^{**}(\mathbf{X})$, $fR(.)$ is called optimal. In other words, if the evaluation of a modal rational function $fR(\mathbf{X})$ is both complete and feasible, $fR(.)$ is optimal for **X**. Optimal functions give tight bounds of complete estimation.

4.1 Uni-incident Optimality

Theorem 4.1 (Armengol *et al*., 2001) If all arguments of $fR(\mathbf{X})$ are uni-incident and they have the same modality,

$$f^*(\mathbf{X}) = fR(\mathbf{X}) = f^{**}(\mathbf{X}).$$

For example, $f(x,y) = (x+y)^2$ is optimal for X = [1,3] and Y = [2,5]. The true range of the function R_f = [9,64]. The natural extension is fR([1,3], [2,5]) = ([1,3] + [2,5])^2 = [9,64]. Similarly, f([3,1], [5,2]) = [64,9] is optimal. However, $g(x,y) = x^2 + 2xy + y^2$ is not optimal. $f(x,y)$ is not optimal for X = [1,3] and Y = [5,2].

4.2 Multi-incident Optimality

Theorem 4.2 (Armengol *et al*., 2001) If $fR(\mathbf{X})$ are totally monotonous for all of its multi-incident arguments, and **XD** is obtained from **X**, by transforming, for every multi-incident component, all incidences into its dual if the corresponding incidence has a mononicity sense contrary to the global one, then

$$f^*(\mathbf{X}) = fR(\mathbf{XD}) = f^{**}(\mathbf{X}).$$

For example, $f(x,y) = xy/(x+y)$ is extended to X = [1,3] and Y = [15,7]. The partial derivatives of f with respect to x and y are all positive within the domain. The partial derivatives of f with respect to the first incidences of x and y are positive, and negative respect to the second incidences of x and y. Therefore,

$$fR(\mathbf{XD}) = [1,3]\times[15,7]/([3,1]+[7,15]) = [0.9375,2.1]$$

is optimal, compared to $gR(X,Y) = 1/(1/X + 1/Y) = 1/(1/[1,3]+1/[15,7]) = [0.9375,2.1]$.

4.3 Example A: true range estimation of one-way clutch

To illustrate the optimality of modal interval in range estimation, a comparison of MIA method and Direct Linearization Method (DLM) (Chase *et al*., 1997) (as implemented in CE/Tol® package) for the one-way clutch example is made, as shown in *Figure 6* and *Table 2*. Compared to the methods of DLM with Root-Sum-Square (RSS) and Worst-Case (WC), MIA gives accurate estimation of true variation range.

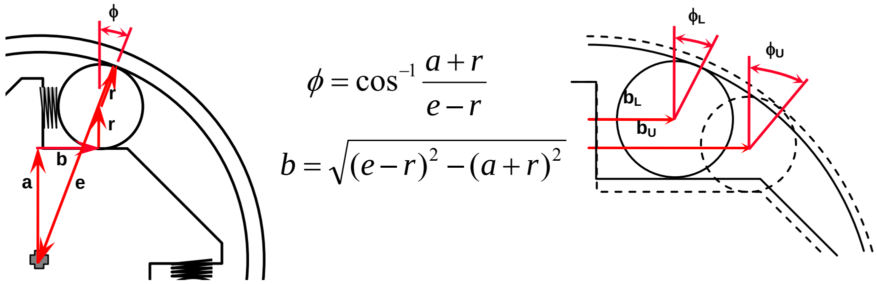


Figure 6. Modal interval makes complex algebraic relations with multi-incident variables interpretable. Interpretations are corresponding to different value sets

Table 2. Result comparison between MIA and DLM method

Input			**Output**: ***position of roller*** (*b*) **True Range** is [**4.0838,5.4405**]		
Hub Height (*a*)	***Ring Radius*** (*e*)	***Roller Radius*** (*r*)	**DLM with Root-Sum-Square** (as in CE/Tol®)	**DLM with Worst-Case** (as in CE/Tol®)	**MIA**
[27.595, 27.695]	[50.7875, 50.8125]	[11.42, 11.44]	[4.3585, 5.2625]	[4.1368, 5.4842]	[4.0838, 5.4405]

4.4 EXAMPLE B: HARD DISK TRACKS

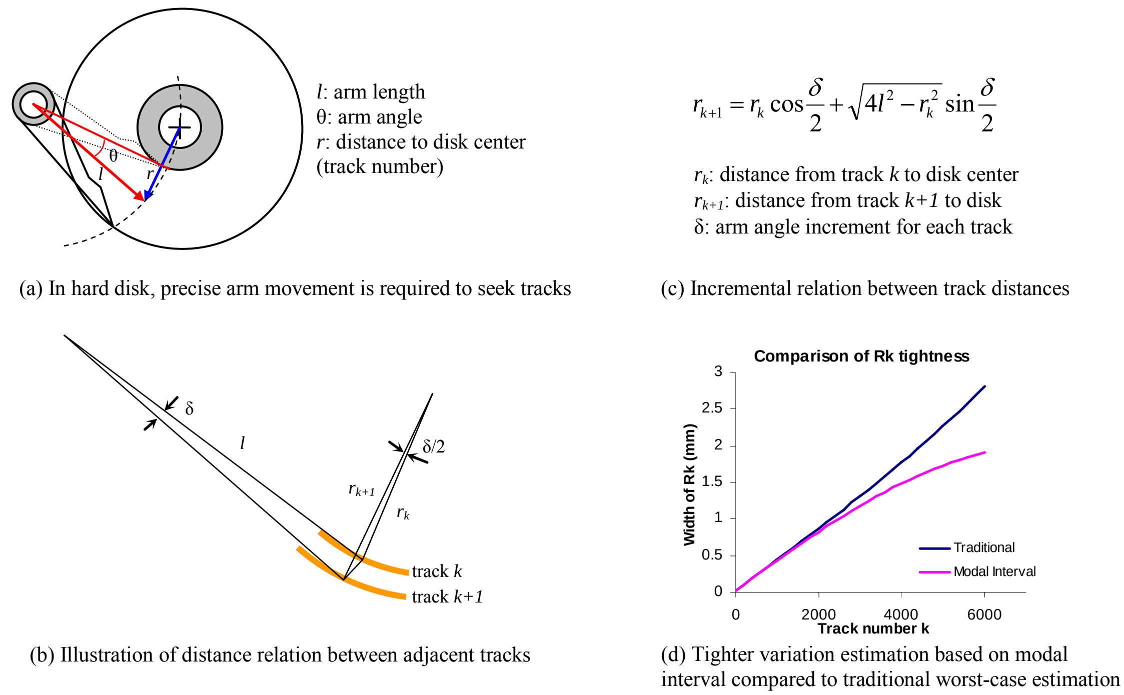
Figure 7 shows an example of hard disk tolerance analysis simulation. To seek tracks, the arm of the hard drive moves certain angles incrementally. Each disk surface may have tens of thousands tracks. Thus, precise movement at high speed is critical to find correct tracks given uncertainty involved in control and geometric variations. A traditional interval model to estimate the distance from each track to disk center is based on

$$R_{k+1} = R_k \cos\frac{\Delta}{2} + \sqrt{4L^2 - R_k^2}\,\sin\frac{\Delta}{2}$$

for L = [42.00, 42.02] mm, Δ = [0.0002, 0.00021], and R_0 = [10.35, 10.37] mm. The over-estimation of the range grows as the track number increases. However, the optimal modal interval model based on

$$R_{k+1} = R_k \cos\frac{\Delta}{2} + \sqrt{4L^2 - [Dual(R_k)]^2}\,\sin\frac{\Delta}{2}$$

gives a tighter range estimate, as compared in Figure 7-d.



(a) In hard disk, precise arm movement is required to seek tracks

(c) Incremental relation between track distances

(b) Illustration of distance relation between adjacent tracks

(d) Tighter variation estimation based on modal interval compared to traditional worst-case estimation

Figure 7. Modal interval gives interpretable and tighter variation estimation result in track distance simulation

4.5 EXAMPLE C: PROCESS CONTROL SIMULATION

A third example of optimality is a derivative process control simulation, which shows the significant difference between modal interval and traditional interval methods, as compared in *Figure 8*. With uncertainty involved in parameters, the tooling speed range estimation with respect to time based on MIA optimal extension

$$V(k+1) = V(k) + K_d[V_0 - dual(V(k))] - \frac{1}{S}[dual(V(k)) - V_a]$$

is much better than that of the worst-case traditional interval extension

$$V(k+1) = V(k) + K_d[V_0 - V(k)] - \frac{1}{S}[V(k) - V_a].$$

$$\frac{dv}{dt} = k_d(v_0 - v) - \frac{1}{s}(v - v_a)$$

v: sensored tooling speed
v_0: nominal control speed
k_d: action factor of controller
v_a: sensor shift due to surroundings
s: sensitivity factor of sensor

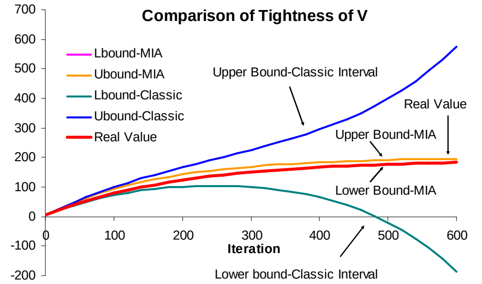
(a) A simple derivative controller model

$$V(k+1) = V(k) + K_d[V_0 - V(k)] - \frac{1}{S}[V(k) - V_a]$$

$$V(k+1) = V(k) + K_d[V_0 - dual(V(k))] - \frac{1}{S}[dual(V(k)) - V_a]$$

$K_d = [0.004, 0.005]$ $V_a = [2,3]$
$S = [1000, 1001]$ $V_0 = [240, 241]$
$V(0) = [3,3]$

(b) Comparison of interval models with classic interval and modal interval



(c) Optimal variation estimation based on modal interval compared to classic interval methods

Figure 8. Modal interval shows optimal estimation of variation in a process control simulation

5 Closed-Loop Tolerance Analysis

Besides the semantic completion described in Section 3, MIA has the good property of structural completion. Traditional set-based interval analysis is not complete. The group properties of addition and multiplication operations are lost. There is no interval $[x,y]$ such that $[a,b]+[x,y]=0$ and the equation $[a,b]+[x,y]=[c,d]$ has an interval solution only when $b-a \le d-c$. For example, $[1,3]+[x,y]=[2,7]$ has solution $[x,y]=[2,7]-[1,3]=[-1,6]$ instead of [1,4]. In contrast, arithmetic operations in MIA are complete.

5.1 Closeness of MIA arithmetic operations

In MIA, it is easy to find true solution for equation $A+X=B$, which is $X=B-dual(A)$, and $AX=B$, which is $X=B/dual(A)$. Thus, $[1,3]+[x,y]=[2,7]$ has the true solution $[x,y]=[2,7]-dual([1,3])=[2,7]-[3,1]=[1,4]$.

Given that a and b have values from intervals [2,4]' and [-2,6]', finding the interval estimation X for the equation $ax=b$ is interpreted as

$$\mathrm{U}(x, X')\mathrm{E}(a,[2,4]')\mathrm{E}(b,[-2,6]')ax=b.$$

Therefore, X will be the proper interval solution of the equation

$$[4,2]\times X = [-2,6].$$

Thus,

$$X = [-2,6]/dual([4,2]) = [-1,3].$$

The optimality of MIA arithmetic allows us to overcome the over-estimation barrier in worse-case stack-up. True range estimation can be achieved without extensive computation as in simulation approach. In addition, the estimated 3D variation vectors from size, geometry, and kinematic tolerances such as the one in *Figure 4* should be *closed* in a complete assembly, that is, tolerance ranges R_i in *x*, *y*, and *z* directions should satisfy $f(R_1, R_2, \cdots, R_n)$ =0. This constraint in turn helps to estimate ranges more accurately. Traditional methods do not consider the closeness constraint. The closeness of MIA arithmetic operations provides the fundamentals for the soundness of semantic tolerancing.

5.2 TOLERANCE ANALYSIS

Tolerance formulation and numerical methods based on MIA arithmetic operations maintain the completeness of interval computation. During the tolerance and kinematic chain formulation, if explicit functions are available in tolerance analysis, such as in Section 4, accurate and interpretable variation ranges can be estimated. If only implicit functions are available, methods to solve modal interval systems are needed.

An interval system of MIA linear equations $\mathbf{A} \cdot \mathbf{X} = \mathbf{B}$, where $\mathbf{A} = (A_{ij})_{n \times n}$ and $\mathbf{B} = (B_i)_{n \times 1}$, is closely associated with two relations $\mathbf{A} \cdot \mathbf{X} \subseteq \mathbf{B}$ and $\mathbf{A} \cdot \mathbf{X} \supseteq \mathbf{B}$.

$$\mathbf{A} \cdot \mathbf{X} = \mathbf{B} \Leftrightarrow \mathbf{A} \cdot \mathbf{X} \subseteq \mathbf{B} \text{ and } \mathbf{A} \cdot \mathbf{X} \supseteq \mathbf{B}.$$

If a *Jacobi interval operator* is defined as

$$\Im(X_i) := \frac{B_i - \sum_{i \neq j} Dual(A_{ij}) \times Dual(X_j)}{Dual(A_{ii})} \quad \left(0 \notin A_{ii} \text{ and } i = 1, \cdots, n\right),$$

the following theorem is the foundation of solving MIA linear systems optimally.

Theorem 5.1 (Sainz et al., 2002a; 2002b) (1) If X is a solution to $\mathbf{A} \cdot \mathbf{X} \subseteq \mathbf{B}$, $\Im(\mathbf{X})$ is a solution to $\mathbf{A} \cdot \mathbf{X} \supseteq \mathbf{B}$. (2) If X is a solution to $\mathbf{A} \cdot \mathbf{X} \supseteq \mathbf{B}$, $\Im(\mathbf{X})$ is a solution to $\mathbf{A} \cdot \mathbf{X} \subseteq \mathbf{B}$.

The Jacobi algorithm to solve MIA linear systems is listed in *Figure 9*. By means of the Jacobi interval operator associated with the linear system **AX** = **B**, it is possible to get a sequence of interval vectors $\mathbf{X}^{(1)} = \Im(\mathbf{X}^{(0)})$, $\mathbf{X}^{(2)} = \Im(\mathbf{X}^{(1)})$, …, which satisfies

$$\mathbf{X}^{(0)} \supseteq \mathbf{X}^{(1)} \subseteq \mathbf{X}^{(2)} \supseteq \cdots \subseteq \mathbf{X}^{(2k)} \supseteq \mathbf{X}^{(2k+1)} \subseteq \cdots,$$

such that $\mathbf{X}^{(2k)}$ is a solution of $\mathbf{A} \cdot \mathbf{X} \supseteq \mathbf{B}$, and $\mathbf{X}^{(2k+1)}$ is a solution of $\mathbf{A} \cdot \mathbf{X} \subseteq \mathbf{B}$.

The Jacobi algorithm does not necessarily converge. The sufficient condition for convergence is described in Theorem 5.2.

Theorem 5.2 (Sainz et al., 2002a) For system **AX** = **B**, if *Prop*(**A**) is a strictly diagonally dominant interval matrix, $\frac{\sum_{j \neq i} Width(A_{ij})}{Width(A_{ii})} < \alpha < 1$, there exists a limit $\mathbf{X}^{\infty}$ satisfying $\mathbf{X}^{\infty} = \Im(\mathbf{X}^{\infty})$.

Input: modal interval matrix **A**, modal interval vector **B**
Output: modal interval vector **X** that satisfies **AX** = **B**

1. Initial estimation of $\mathbf{Y}^{(0)}$ such that $Impr(\mathbf{A}) \cdot \mathbf{Y}^{(0)} \subseteq Prop(\mathbf{B})$;
2. $\mathbf{X}^{(0)} = \Im(\mathbf{Y}^{(0)})$, which is the initial solution for $\mathbf{A} \cdot \mathbf{X} \supseteq \mathbf{B}$;
3. Iterate the follows for *p* times: $\mathbf{X}^{(t)} = \Im(\mathbf{X}^{(t-1)})$.

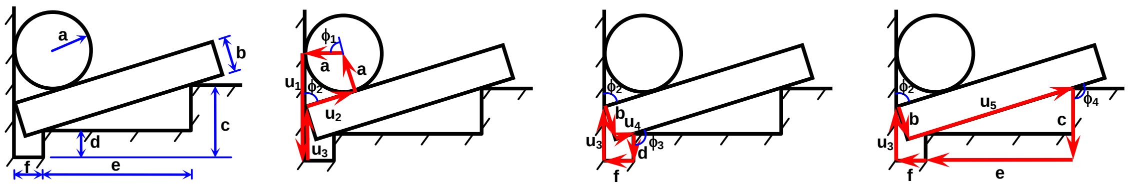
Figure 9. Jacobi algorithm to solve linear systems of modal intervals [**Error! Bookmark not defined.**]

If A is not strictly diagonally dominant, general interval methods such as in references (Neumaier, 1990; Hansen, 1992; Ning and Kearfott, 1997) can be used to solve interval linear equations. However, the interpretability is compromised.

When variation functions are nonlinear, a linearization process may be used to reduce the complexity of direct computation of nonlinear functions. This linear approximation changes semantics relation between variables. Again, as a result of linearization, the tolerance interpretability and optimality principles generally do not apply to the numerical results.

5.3 EXAMPLE D: STACKED BLOCK ASSEMBLY – NONLINEAR

A closed vector loop defines relations among size, geometry, and kinematic variations. The sum of vector components in each translational or rotational direction should be equal to zero. *Figure 10* shows an example of stacked block assembly. With known size tolerances, the kinematic variation of the stacked blocks can be calculated with three loops. The parameter values and formulation of loops are listed in *Table 3*.



(a) known parameters (b) vector loop 1 (c) vector loop 2 (d) vector loop 3

Figure 10. Stacked block assembly

Table 3. The variation formulation of loops

Known size variation	$a = 6.62 \pm 0.2$ $b = 6.805 \pm 0.075$ $c = 10.675 \pm 0.125$ $d = 4.06 \pm 0.15$ $e = 24.22 \pm 0.35$ $f = 3.905 \pm 0.125$
Unknown kinematic variation	$u_1 = 18.7181 \pm ?$ $u_2 = 8.6705 \pm ?$ $u_3 = 10.0477 \pm ?$ $u_4 = 2.1894 \pm ?$ $u_5 = 27.2965 \pm ?$ $\phi_1 = 74.7243 \pm ?$ $\phi_2 = -74.7243 \pm ?$ $\phi_3 = -105.2761 \pm ?$ $\phi_4 = -105.2761 \pm ?$
Loop 1	$\begin{cases} F_1 = u_2\cos(90+\phi_2) + a\cos(180+\phi_2) + a\cos(180+\phi_1+\phi_2) = 0 \\ F_2 = u_3 + u_2\sin(90+\phi_2) + a\sin(180+\phi_2) + a\sin(180+\phi_1+\phi_2) - u_1 = 0 \\ F_3 = 90+\phi_2+90+\phi_1+90+90-360 = 0 \end{cases}$
Loop 2	$\begin{cases} F_4 = b\cos(\phi_2) + u_4\cos(\phi_2+90) + d\cos(\phi_2+90+\phi_3) - f = 0 \\ F_5 = u_3 + b\sin(\phi_2) + u_4\sin(\phi_2+90) + d\sin(\phi_2+90+\phi_3) = 0 \\ F_6 = 90+\phi_2-90+90+\phi_3-90+180 = 0 \end{cases}$
Loop 3	$\begin{cases} F_7 = b\cos(\phi_2) + u_5\cos(\phi_2+90) + c\cos(\phi_2+90+\phi_4) - e - f = 0 \\ F_8 = u_3 + b\sin(\phi_2) + u_5\sin(\phi_2+90) + c\sin(\phi_2+90+\phi_4) = 0 \\ F_9 = 90+\phi_2-90+90+\phi_4-90+180 = 0 \end{cases}$

To solve nonlinear functions $F(\mathbf{s},\mathbf{k}) = 0$, where s is a size variation vector and k is a kinematic variation vector, linearization process by Taylor's expansion

$$\left[\frac{\partial F_i}{\partial s_j}\right]_{i\times j} \Delta\mathbf{s} + \left[\frac{\partial F_i}{\partial k_i}\right]_{i\times i} \Delta\mathbf{k} = 0$$

with respect to nominal values is conducted before interval linear method is used to estimate the variation. The results from the MIA linearization method and the DLM method (Chase *et al*., 1997) are compared in *Table 4*.

Table 4. Comparison of MIA linearization and DLM

MIA Linearization	DLM Worst-Case	DLM Statistical
$\Delta u_1 = [0.5420,\ -0.5420]$	$\Delta u_1 = [-0.5421,\ 0.5421]$	$\Delta u_1 = [-0.2998,\ 0.2998]$
$\Delta u_2 = [0.4672,\ -0.4672]$	$\Delta u_2 = [-0.3899,\ 0.3899]$	$\Delta u_2 = [-0.2725,\ 0.2725]$
$\Delta u_3 = [0.3137,\ -0.3137]$	$\Delta u_3 = [-0.2942,\ 0.2942]$	$\Delta u_3 = [-0.1844,\ 0.1844]$
$\Delta u_4 = [0.2729,\ -0.2729]$	$\Delta u_4 = [-0.2384,\ 0.2384]$	$\Delta u_4 = [-0.1411,\ 0.1411]$
$\Delta u_5 = [0.5209,\ -0.5209]$	$\Delta u_5 = [-0.5174,\ 0.5174]$	$\Delta u_5 = [-0.3836,\ 0.3836]$
$\Delta \phi_1 = [0.0228,\ -0.0228]$	$\Delta \phi_1 = [-0.8156,\ 0.8156]$	$\Delta \phi_1 = [-0.4784,\ 0.4784]$
$\Delta \phi_2 = [0.0228,\ -0.0228]$	$\Delta \phi_2 = [-0.8156,\ 0.8156]$	$\Delta \phi_2 = [-0.4784,\ 0.4784]$
$\Delta \phi_3 = [0.0228,\ -0.0228]$	$\Delta \phi_3 = [-0.8156,\ 0.8156]$	$\Delta \phi_3 = [-0.4784,\ 0.4784]$
$\Delta \phi_4 = [0.0228,\ -0.0228]$	$\Delta \phi_4 = [-0.8156,\ 0.8156]$	$\Delta \phi_4 = [-0.4784,\ 0.4784]$

5.4 EXAMPLE E: STACKED BLOCK ASSEMBLY – LINEAR

Suppose that the limits of angle variation in previous stacked block assembly example are known, the tolerance analysis problem is then reduced to linear equation solving. This linear problem can be solved using the Jocobi algorithm, and the result is interpretable, as listed in *Table 5*.

Table 5. Linear problem in stacked block assembly

Known size variation	$a = 6.62 \pm 0.2$ $\quad b = 6.805 \pm 0.075$ $\quad c = 10.675 \pm 0.125$ $d = 4.06 \pm 0.15$ $\quad e = 24.22 \pm 0.35$ $\quad f = 3.905 \pm 0.125$
Known kinematic variation	$\phi_1 = 74.7243 \pm 0.4281$ $\quad \phi_2 = -74.7243 \pm 0.4281$ $\phi_3 = -105.2761 \pm 0.4281$ $\quad \phi_4 = -105.2761 \pm 0.4281$
Unknown kinematic variation	$u_1 = 18.7181 \pm ?$ $\quad u_2 = 8.6705 \pm ?$ $\quad u_3 = 10.0477 \pm ?$ $\quad u_4 = 2.1894 \pm ?$ $\quad u_5 = 27.2965 \pm ?$
Linear equations	$\begin{cases} -u_1 + u_2 \sin(90+\phi_2) + u_3 + a\sin(180+\phi_2) = 0 \\ u_2 \cos(90+\phi_2) + a\cos(180+\phi_2) - a = 0 \\ u_3 + u_4 \sin(\phi_2+90) + b\sin(\phi_2) - d = 0 \\ u_4 \cos(\phi_2+90) + b\cos(\phi_2) - f = 0 \\ u_5 \cos(\phi_2+90) + b\cos(\phi_2) - e - f = 0 \end{cases}$

	$\begin{bmatrix} -1 & [0.2562,0.2707] & 1 & 0 & 0 \\ 0 & [0.9626,0.9667] & 0 & 0 & 0 \\ 0 & 0 & 1 & [0.2562,0.2707] & 0 \\ 0 & 0 & 0 & [0.9626,0.9667] & 0 \\ 0 & 0 & 0 & 0 & [0.9626,0.9667] \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{bmatrix} = \begin{bmatrix} [-6.1804,-6.5922] \\ [8.6659,8.0652] \\ [10.8602,10.3888] \\ [2.3054,1.9179] \\ [26.8754,25.7879] \end{bmatrix}$
Result of Jacobi algorithm	$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{bmatrix} = \begin{bmatrix} [18.7024,18.7335] \\ [9.0026,8.34302] \\ [10.2466,9.85174] \\ [2.39497,1.98397] \\ [27.9196,26.6762] \end{bmatrix}$
Interpretation of result	$U(u_1,[18.7024,18.7335]')U(a,[6.42,6.82]')U(b,[6.73,6.88]')U(c,[10.55,10.8]')U(d,[3.91,4.21]')$ $U(e,[23.87,24.57]'U(f,[3.78,4.03]')U(\phi_1,[74.2962,75.1524]')U(\phi_2,[-75.1524,-74.2962]')$ $U(\phi_3,[-105.7042,-104.8480]')U(\phi_4,[-105.7042,-104.8480]')$ $E(u_2,[8.34302,9.0026]')E(u_3,[9.85174,10.2466]')E(u_4,[1.98397,2.39497]')E(u_5,[26.6762,27.9196]$ $\begin{cases} -u_1 + u_2\sin(90+\phi_2) + u_3 + a\sin(180+\phi_2) = 0 \\ u_2\cos(90+\phi_2) + a\cos(180+\phi_2) - a = 0 \\ u_3 + u_4\sin(\phi_2+90) + b\sin(\phi_2) - d = 0 \\ u_4\cos(\phi_2+90) + b\cos(\phi_2) - f = 0 \\ u_5\cos(\phi_2+90) + b\cos(\phi_2) - e - f = 0 \end{cases}$

6 Conclusion

A semantic tolerance modeling scheme based on modal interval is proposed to enrich the modeling and analysis structure for tolerances such that tolerancing semantics can be embedded in mathematic representation in order to support better design and manufacturing specifications.

The new semantic tolerancing method captures engineering and logic relation between specifications and prevents the degeneracy of engineering semantics during mathematic calculation. Priori and posteriori variations in tolerance specification are differentiated. The model captures the semantics of physical property difference between rigid and flexible materials as well as tolerancing intents such as sequence of specification, measurement, and assembly. Compared to traditional methods, the semantic tolerancing allows us to estimate true variation ranges such that feasible and complete solutions can be obtained.

Future research may include tolerance chain formulation with the consideration of geometric tolerances and interaction between tolerances, optimization approach to solve linear and nonlinear modal interval equations, as well as tolerance synthesis based on global optimization methods of interval analysis.

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