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Closed-Loop Analysis in Semantic Tolerance Modeling

A semantic tolerance modeling scheme based on generalized intervals was recently proposed to allow for embedding more tolerancing intents in specifications with a combination of numerical intervals and logical quantifiers. By differentiating a priori and a posteriori tolerances, the logic relationships among variables can be interpreted, which is useful to verify completeness and soundness of numerical estimations in tolerance analysis. In this paper, we present a semantic tolerance analysis approach to estimate size and geometric tolerance stack-ups based on closed loops of interval vectors. An interpretable linear system solver is constructed to ensure interpretability of numerical results. A direct linearization method for nonlinear systems is also developed. This new approach enhances traditional numerical analysis methods by preserving logical information during computation such that more semantics can be derived from variation estimations.

1. Introduction

In tolerance analysis, estimations of accumulative tolerances are mathematically formulated and solved in different ways. The typical approaches include variational estimation, kinematic formulation, statistical approximation, and Monte Carlo simulation. The analysis process is simplified to the computation of pure numerical intervals. Methods of linearization and high-order Taylor approximations are extensively used to compute parameters (e.g., statistical moments) and variables (e.g., kinematic variations in assemblies). Because of these numerical treatments, completeness and soundness of range estimations are compromised. A *complete* solution includes all possible occurrences, which is to check if the range estimation includes all possible stack-up results. Conversely, a *sound* solution does not include impossible occurrences, which consists in checking if the interval overestimates the actual range.

The traditional worst-case linear stack-up methods focus on completeness while range estimations may not be sound. The results usually are overly pessimistic. In contrast, Monte Carlo methods focus on soundness while estimations may not be complete. Assuming the applied distributions and their parameters reflect the true variations, the simulated ranges are complete only when the sample size is enormously large such that the pseudo-random numbers from a full-period random number generator are exhausted. Kinematic formulation methods may result in solutions that are neither complete nor sound because of numerical treatments. This is illustrated by an example of one-way clutch in Figure 1. The known dimensional tolerances are the hub height $\mathbf{a}=[27.595,27.695]$, the ring radius $\mathbf{e}=[50.7875,50.8125]$, and the roller radius $\mathbf{r}=[11.42,11.44]$. The variation of the roller position b needs to be estimated. By the direct linearization methods (DLM) with root-sum-square (RSS) and worst-case (WC) [1], we have the estimations $\mathbf{b}^{RSS}=[4.3585,5.2625]$ and $\mathbf{b}^{WC}=[4.1368,5.4842]$ respectively. However, the true variation range is $\mathbf{b}=[4.0838,5.4405]$, which can be derived from the direct analysis of geometry. The combination of the largest *a* and *r* and the smallest *e* generates the lower bound of *b*. The combination of the smallest *a* and *r* and the largest *e* forms the upper bound of *b*. We can see $\mathbf{b}^{RSS}$ is sound but not complete, whereas $\mathbf{b}^{WC}$ is neither complete nor sound.

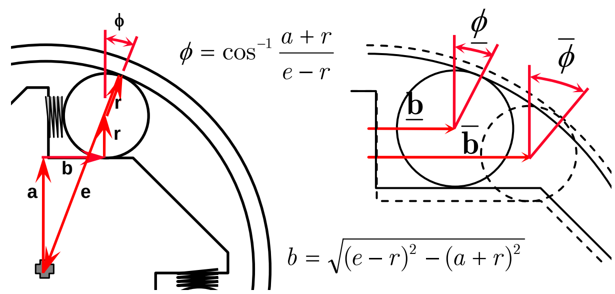


Figure 1. An example of one-way clutch variation estimation

Let $z=f(x_1,\ldots,x_n)$ be a general relation in tolerance analysis, where x_i's ($i=1,\ldots,n$) are the variation source variables (inputs), and z is the performance variable (output). Let $[\underline{x}_i,\overline{x}_i]$'s ($i=1,\ldots,n$) be the respective intervals of the input tolerances and $[\underline{z},\overline{z}]$ a variation range estimate. $[\underline{z},\overline{z}]$ is complete if and only if the following statement is true: "for any combination of inputs x_i's within the respective $[\underline{x}_i,\overline{x}_i]$'s, the output $z=f(x_1,\ldots,x_n)$ must be included in the estimated $[\underline{z},\overline{z}]$". That is,

$$(\forall x_1\in[\underline{x}_1,\overline{x}_1])\cdots(\forall x_n\in[\underline{x}_n,\overline{x}_n])(\exists z\in[\underline{z},\overline{z}])(f(x_1,\ldots,x_n)=z)$$

Similarly, the estimation is sound if and only if the following statement is true: "for any output z within the estimated $[\underline{z},\overline{z}]$, there must exist a combination of inputs x_i's within the respective $[\underline{x}_i,\overline{x}_i]$'s such that $z = f(x_1,\ldots,x_n)$". That is,

$$(\forall z \in [\underline{z},\overline{z}])(\exists x_1 \in [\underline{x}_1,\overline{x}_1])\cdots(\exists x_n \in [\underline{x}_n,\overline{x}_n])(f(x_1,\ldots,x_n) = z)$$

For instance, in the one-way clutch example of Figure 1, we are able to assert

$$(\forall a \in [27.595,27.695])(\forall e \in [50.7875,50.8125])$$
$$(\forall r \in [11.42,11.44])(\exists b \in [4.0838,5.4405])$$
$$\left(b = \sqrt{(e-r)^2 + (a+r)^2}\right)$$

and

$$(\forall b \in [4.0838,5.4405])(\exists a \in [27.595,27.695])$$
$$(\exists e \in [50.7875,50.8125])(\exists r \in [11.42,11.44])$$
$$\left(b = \sqrt{(e-r)^2 + (a+r)^2}\right)$$

Therefore, the logic interpretation of numerical results as above enables us to assess the completeness and soundness of range estimations. The attention of interpretability needs to be given in tolerance analysis. Recently, we proposed a new semantic tolerance modeling scheme [2, 3, 4, 5] based on generalized intervals to enhance the interpretability of tolerance modeling. The purpose of semantic tolerance modeling is to embed logic relationships and engineering implications into the mathematical representation. With logical quantifiers, the relationship between tolerance specifications and implications of stacking may be derived from formulations. With the explicit differentiation between a priori and a posteriori tolerances, models can capture process-oriented semantics such as the difference between rigid and flexible materials in assemblies and the sequence of assembly.

In this paper, we present a tolerance analysis approach based on interval vector loops to estimate semantic tolerance accumulations. To ensure the interpretability of numerical results, interpretable Jacobi algorithms are developed to solve interval linear systems. Based on the algebraic closure property, we can formulate constrained problems with closed loops of interval vectors. Geometric tolerances can also be included in the loops with the consideration of interdependency between size and geometric tolerances. In the remainder of the paper, a brief review of vector loop based tolerance analysis methods and the generalized interval as the basis of semantic tolerance modeling are given in Section 2. Section 3 presents the proposed analysis approach for semantic tolerances. An interpretable linear system solver to ensure interpretability is constructed. The new approach is illustrated with examples. Section 4 describes the closed-loop approach to integrate geometric tolerances.

2. Background

There is a substantial amount of literature on tolerance modeling, analysis, and synthesis [6, 7]. Here, we only give a brief overview of vector loop based analysis methods that are closely related to the proposed closed-loop semantic tolerance analysis approach, as reviewed in Section 2.1. The main properties and notations of generalized intervals are summarized in Section 2.2.

2.1 Vector Loop based Tolerance Analysis

Traditionally tolerance analysis is product-oriented. Dimensional limit, geometric variation, and kinematic displacement can be modeled mathematically in vectors and matrices. The vectorial tolerancing methods (Wirtz *et al*. [8], Martinsen [9]) model size, form, location, and orientation tolerances in a unified vector format in order to provide an integrated quality control loop. Rivest *et al*. [10] employed the kinematic characteristics of links between datum and toleranced features to model chains of variations. Clément *et al.* [11] identified and analyzed functional elements called TTRSs which are associated with geometric constraints. The small-displacement torsor methods (Bourdet and Ballot [12], Giordano and Duret [13], Descrochers [14]) approximate the rotation and translation displacement in the form of torsors. The matrix representation methods (Whitney *et al*. [15], Desrochers and Riviere [16], Lafond and Laperrière [17]) model small displacements in kinematic chains in the form of homogenous transformation matrices. Recently, Desrochers *et al.* [18] combined the torsor and matrix-based representations for tolerance analysis. Chase *et al*. [1, 19, 20] performed analysis of assemblies with tolerance vectors and small kinematic adjustments with linear approximations of implicit geometric constraints. Sacks and Joskowicz [21] analyzed 2D kinematic tolerances of assemblies with contact changes by the aid of contact constraints. Zou and Morse [22] proposed a fitting condition test method based on geometric constraints of gap closure between components.

In recent years, process-oriented analysis approaches were also proposed to consider the accumulation effects of manufacturing processes. With 1D vector loops, Zhang [23] combined the relation between functional requirements and dimensional tolerances with the one between dimensional and machining tolerances for simultaneous tolerancing. Based on constraints of force closure (Liu and Hu [24], Chang and Gossard [25]), 3D vector loops were used to predict variation accumulation in sheet metal joining with the linearized finite element formulation. Long and Hu [26] extended the method to include the variation of fixtures during assembly operations. The single-station methods were also extended to multi-station approaches (Shiu *et al.* [27], Camelio *et al.* [28]) where variations are propagated in stages with tooling variations incorporated. Recently, Huang *et al.* [29, 30] developed a stream-of-variation method to estimate dimensional variations in rigid-body assemblies for single-station and multi-station systems considering fixtures based on kinematic constraints.

In the above vector loop based methods, variation problems are formulated based on constraints of either form closure or force closure. The numerical treatments applied in these approaches prohibit interpretable numerical results. The main reason is that the commonly used solving methods with linearization and high-order approximations do not incorporate interpretability. During computation, the logic relationships among variables are left out. Therefore, the completeness and soundness of the results cannot be verified. In this paper, we propose a semantic tolerance analysis approach based on a new structure of interpretable linear system solver. Generalized intervals are used for a unified variation representation.

2.2 Generalized Intervals

The semantic tolerance model is based on modal interval analysis (MIA) [31, 32, 33], which is an algebraic and semantic extension of the classic interval analysis (IA) [34]. A *modal* *interval* or *generalized interval* $\mathbf{x} := [\underline{x}, \overline{x}] \in \mathbb{KR}$ is called *proper* when $\underline{x} \le \overline{x}$ and *improper* when $\underline{x} \ge \overline{x}$. The set of proper intervals is denoted by $\mathbb{IR} = \{[\underline{x}, \overline{x}] \mid \underline{x} \le \overline{x}\}$ and the set of improper intervals by $\overline{\mathbb{IR}} = \{[\underline{x}, \overline{x}] \mid \underline{x} \ge \overline{x}\}$. The width of $\mathbf{x}$ is $\mathrm{wid}[\underline{x}, \overline{x}] := |\overline{x} - \underline{x}|$, and the center is found by $\mathrm{mid}[\underline{x}, \overline{x}] := \frac{\underline{x} + \overline{x}}{2}$. A real function $f(x)$ where $x \in \mathbb{R}^n$ can be extended to $\mathbf{f}(\mathbf{x})$ where $\mathbf{x} \in \mathbb{KR}^n$, which is called a $\mathbb{KR}$-extension, AE-extension, or modal extension. The real arithmetic is extended to the so-called Kaucher arithmetic [35].

Three special operators, *pro*, *imp*, and *dual*, are defined in the Kaucher arithmetic. Given a generalized interval $\mathbf{x} = [\underline{x}, \overline{x}] \in \mathbb{KR}$, $\mathrm{pro}\,\mathbf{x} := [\min(\underline{x}, \overline{x}), \max(\underline{x}, \overline{x})]$ and $\mathrm{imp}\,\mathbf{x} := [\max(\underline{x}, \overline{x}), \min(\underline{x}, \overline{x})]$ return the respective proper and improper interval values. $\mathrm{dual}[\underline{x}, \overline{x}] := [\overline{x}, \underline{x}]$ builds a relationship between proper and improper intervals. Related to the arithmetic operations $\circ \in \{+, -, \times, \div\}$, $(\mathrm{dual}\,\mathbf{x}) \circ (\mathrm{dual}\,\mathbf{y}) = \mathrm{dual}(\mathbf{x} \circ \mathbf{y})$. The *inclusion* relationship between modal intervals is defined as $[\underline{x}, \overline{x}] \subseteq [\underline{y}, \overline{y}] \Leftrightarrow (\underline{y} \le \underline{x}) \wedge (\overline{x} \le \overline{y})$. The *less than or equal to* relationship is defined as $[\underline{x}, \overline{x}] \le [\underline{y}, \overline{y}] \Leftrightarrow (\underline{x} \le \underline{y}) \wedge (\overline{x} \le \overline{y})$.

Table 1 lists the major differences between MIA and IA. Different from IA, the group property is maintained in MIA because generalized intervals are closed under the Kaucher arithmetic operations. A generalized interval $\mathbf{a}$ is an *algebraic solution* of the equation $\mathbf{f}(\mathbf{x}) = \mathbf{b}$ where $\mathbf{x}$ is unknown if the original algebraic relation is still valid when the variable $\mathbf{x}$ is replaced by the interval result $\mathbf{a}$, i.e., $\mathbf{f}(\mathbf{a}) = \mathbf{b}$. This property is called *algebraic closure*, which is not available in IA. The group properties under addition and multiplication are lost in IA. For example, the solution of $[1,3] + \mathbf{x}' = [2,7]$ is $\mathbf{x}' = [2,7] - [1,3] = [-1,6]$ in IA. However, if the solution is substituted back to the original equation, $[1,3] + [-1,6] = [0,9] \ne [2,7]$. Therefore $\mathbf{x}'$ is not an algebraic solution. The multiplication and division operators are similar. In contrast, in MIA the solution of $\mathbf{a} + \mathbf{x} = \mathbf{b}$ is $\mathbf{x} = \mathbf{b} - \mathrm{dual}\,\mathbf{a}$; and the solution of $\mathbf{a}\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = \mathbf{b} / \mathrm{dual}\,\mathbf{a}$ when $0 \notin \mathrm{pro}\,\mathbf{a}$. For example, $[1,3] + \mathbf{x} = [2,7]$ has the algebraic solution $\mathbf{x} = [2,7] - \mathrm{dual}[1,3] = [2,7] - [3,1] = [1,4]$ since $[1,3] + [1,4] = [2,7]$. The algebraic closure property is the basis of our closed-loop analysis scheme. It simplifies the numerical analysis process while interpretability is preserved. The numerical interval results always satisfy the original constraints of form closure. Therefore, we call our scheme closed-loop.

Another uniqueness of generalized intervals is the modal semantic extension. Unlike IA which identifies an interval by a set of real numbers only, MIA identifies an interval by a set of predicates which is fulfilled by real numbers. Each interval $\mathbf{x} \in \mathbb{KR}$ has an associated logical quantifier, either existential ($\exists$) or universal ($\forall$). For a real relation $\phi(x) = z$ where $x \in \mathbb{R}^n$ and $z \in \mathbb{R}$, the semantics of its modal extension can be expressed with quantifiers, which are derived based on the modalities of generalized intervals. As universal quantifiers precede existential ones, such quantified propositions have the form of

$$(\forall x_{\mathcal{P}} \in \mathbf{x}_{\mathcal{P}})(\mathrm{Q}_z\, z \in \mathrm{pro}\,\mathbf{z})(\exists x_I \in \mathrm{pro}\,\mathbf{x}_I)(\phi(x) = z)$$

where $\mathcal{P}$ and I are disjoint sets of indices for proper and improper components of $\mathbf{x}_{\mathcal{P} \cup I} \in \mathbb{KR}^n$, $\mathrm{Q}_z = \forall$ if $\mathbf{z} \in \overline{\mathbb{IR}}$, and $\mathrm{Q}_z = \exists$ if $\mathbf{z} \in \mathbb{IR}$.

Table 1. The major differences between MIA and traditional IA

	Classic Interval Analysis	**Modal Interval Analysis**
Validity	$[3,2]$ is an invalid or empty interval	Both $[2,3]$ and $[3,2]$ are valid intervals
Semantics richness	$[2,3] + [2,4] = [4,7]$ is the only valid relation for +, and it only means "stack-up" and "worst-case". $-, \times, /$ are similar.	$[2,3] + [2,4] = [4,7]$, $[2,3] + [4,2] = [6,5]$, $[3,2] + [2,4] = [5,6]$, $[3,2] + [4,2] = [7,4]$ are all valid, and each has a different meaning. $-, \times, /$ have similar semantic properties.
Group property	$\mathbf{a} + \mathbf{x} = \mathbf{b}$, but $\mathbf{x} \ne \mathbf{b} - \mathbf{a}$ $[2,3] + [2,4] = [4,7]$, $[2,4] \ne [4,7] - [2,3]$ $\mathbf{a} \times \mathbf{x} = \mathbf{b}$, but $\mathbf{x} \ne \mathbf{b}/\mathbf{a}$ $[2,3] \times [3,4] = [6,12]$, $[3,4] \ne [6,12]/[2,3]$ $\mathbf{x} - \mathbf{x} \ne 0$ $[2,3] - [2,3] = [-1,1] \ne 0$	$\mathbf{a} + \mathbf{x} = \mathbf{b}$, and $\mathbf{x} = \mathbf{b} - \mathrm{dual}\,\mathbf{a}$ $[2,3] + [2,4] = [4,7]$, $[2,4] = [4,7] - [3,2]$ $\mathbf{a} \times \mathbf{x} = \mathbf{b}$, and $\mathbf{x} = \mathbf{b}/\mathrm{dual}\,\mathbf{a}$ $[2,3] \times [3,4] = [6,12]$, $[3,4] = [6,12]/[3,2]$ $\mathbf{x} - \mathrm{dual}\,\mathbf{x} = 0$ $[2,3] - [3,2] = 0$

3. Closed-Loop Tolerance Analysis

The purpose of semantic tolerance model is to enrich tolerance modeling and analysis structures such that more process-oriented tolerancing semantics and intents can be embedded in mathematical representations. Interpretability is useful to verify completeness and soundness of interval results. Thus interpretable relations among variables should be maintained during computation. In this section, we describe the new interpretable linear system solver to ensure interpretability. At the same time, the algebraic closure of generalized intervals keeps the numerical computation simple enough. We formulate the form closure constraints of small displacement with closed loops of interval vectors. The new approach enhances numerical analysis methods by ensuring algebraic closure and interpretability.

In closed-loop tolerance analysis, the interval vectors that represent size, geometry, and kinematic variations form closed loops in the 3D Euclidean space. That is, the variations $\mathbf{v}_i \in \mathbb{KR}$ in each of x, y, and z directions should have the algebraic relations $\mathbf{f}(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n) \subseteq 0$ or $\mathbf{f}(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n) \supseteq 0$. To form closed-loop tolerance chains, a priori tolerances with the semantics of universal and a posteriori tolerances with the semantics of existential need to be explicitly differentiated. A posteriori variations provide "buffers" in tolerance allocation to make algebraic relations valid and close the loop. If the traditional tolerancing without the differentiation of a priori and a posteriori tolerances is regarded as "passive" tolerancing, semantic tolerancing is "active" tolerancing so as to close the loops of tolerance chains. In general, a priori tolerances are tolerances with predetermined variations. They have the semantics of uncontrollable, unchangeable, critical, hard-constrained, specified, etc. A posteriori tolerances are those with derived variations. They have the semantics of controllable, adjustable, flexible, soft-constrained, feedback, etc.

Tolerance formulation and analysis methods based on generalized intervals and Kaucher arithmetic maintain the algebraic closure of interval computation. During the tolerance and kinematic chain formulation, if explicit functions are available to estimate variations of assemblies, accurate and interpretable ranges can be estimated based on the interpretability and optimality principles [5]. If only implicit functions are available, methods to solve generalized interval systems are needed. In Section 3.1, we describe the new interpretable linear system solver that preserves interpretable relationships. The algorithms and the advantage of interpretability are illustrated with an example in Section 3.2. In Section 3.3, a MIA direct linearization method is presented to solve nonlinear problems. A nonlinear example is given in Section 3.4.

3.1 Solving interpretable linear systems

As mentioned in Section 2.1, the linearization approach used in the existing vector loop based analysis methods does not support interpretability. Thus the completeness and soundness of the numerical results cannot be verified. Here, we describe a new linearization and solving process that generates interpretable numerical results.

For $\mathbf{x} \in \mathbb{KR}^n$, a linear system of generalized intervals

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{b} \tag{1}$$

where $\mathbf{A} = (\mathbf{a}_{ij})_{n\times n} \in \mathbb{KR}^{n\times n}$ and $\mathbf{b} \in \mathbb{KR}^n$, is closely associated with two inclusion relationships $\mathbf{A}\cdot\mathbf{x} \subseteq \mathbf{B}$ and $\mathbf{A}\cdot\mathbf{x} \supseteq \mathbf{B}$, given as

$$\mathbf{A}\cdot\mathbf{x} = \mathbf{B} \Leftrightarrow (\mathbf{A}\cdot\mathbf{x} \subseteq \mathbf{B}) \wedge (\mathbf{A}\cdot\mathbf{x} \supseteq \mathbf{B}) \tag{2}$$

If a *Jacobi interval operator* is defined as

$$\Im(\mathbf{x}_i) := \frac{\mathbf{b}_i - \sum_{i\neq j} \operatorname{dual}\mathbf{a}_{ij} \cdot \operatorname{dual}\mathbf{x}_j}{\operatorname{dual}\mathbf{a}_{ii}} \quad (0 \notin \operatorname{pro}\mathbf{a}_{ii} \text{ and } i = 1, \cdots, n) \tag{3}$$

the following theorem provides the foundation to solve the linear system in Eq.(1).

Theorem 3.1 [36] (1) If $\mathbf{x}$ is a solution to $\mathbf{A}\cdot\mathbf{x} \subseteq \mathbf{b}$, $\Im(\mathbf{x})$ is a solution to $\mathbf{A}\cdot\mathbf{x} \supseteq \mathbf{b}$. (2) If $\mathbf{x}$ is a solution to $\mathbf{A}\cdot\mathbf{x} \supseteq \mathbf{b}$, $\Im(\mathbf{x})$ is a solution to $\mathbf{A}\cdot\mathbf{x} \subseteq \mathbf{b}$.

However, the linear system in Eq.(1) is not interpretable if it includes multi-incident $\mathbf{x}_j$'s which are existential. That is, a variable $\mathbf{x}_j$ appears multiple times in the equation. Because the concatenation of $\forall x \in \mathbf{x}$ and $\forall x \in \mathbf{x}$ is $\forall x \in \mathbf{x}$, and the concatenation of $\forall x \in \mathbf{x}$ and $\exists x \in \mathbf{x}$ is $\exists x \in \mathbf{x}$. But the concatenation of $\exists x \in \mathbf{x}$ and $\exists x \in \mathbf{x}$ is not $\exists x \in \mathbf{x}$ in general. Formally

$$(\mathrm{Q}_{y_1} y_1 \in \operatorname{pro}\mathbf{y}_1)(\exists x \in \operatorname{pro}\mathbf{x})(\mathrm{Q}_{z_1} z_1 \in \operatorname{pro}\mathbf{z}_1)(z_1 = f_1(x, y_1))$$

and

$$(\mathrm{Q}_{y_2} y_2 \in \operatorname{pro}\mathbf{y}_2)(\exists x \in \operatorname{pro}\mathbf{x})(\mathrm{Q}_{z_2} z_2 \in \operatorname{pro}\mathbf{z}_2)(z_2 = f_2(x, y_2))$$

do not necessarily lead to

$$(\mathrm{Q}_{y_1} y_1 \in \operatorname{pro}\mathbf{y}_1)(\mathrm{Q}_{y_2} y_2 \in \operatorname{pro}\mathbf{y}_2)(\exists x \in \operatorname{pro}\mathbf{x})(\mathrm{Q}_{z_1} z_1 \in \operatorname{pro}\mathbf{z}_1)$$
$$(\mathrm{Q}_{z_2} z_2 \in \operatorname{pro}\mathbf{z}_2)(z_1 = f_1(x, y_1) \wedge z_2 = f_2(x, y_2))$$

To ensure interpretability, a transformed and interpretable linear system

$$\begin{cases} \mathbf{a}_{11}\mathbf{x}_1 + \mathbf{a}_{12}\operatorname{pro}\mathbf{x}_2 + \ldots + \mathbf{a}_{1n}\operatorname{pro}\mathbf{x}_n \subseteq \mathbf{b}_1 \\ \mathbf{a}_{21}\operatorname{pro}\mathbf{x}_1 + \mathbf{a}_{22}\mathbf{x}_2 + \ldots + \mathbf{a}_{2n}\operatorname{pro}\mathbf{x}_n \subseteq \mathbf{b}_2 \\ \ldots \\ \mathbf{a}_{n1}\operatorname{pro}\mathbf{x}_1 + \mathbf{a}_{n2}\operatorname{pro}\mathbf{x}_2 + \ldots + \mathbf{a}_{nn}\mathbf{x}_n \subseteq \mathbf{b}_n \end{cases} \tag{4}$$

or

$$\begin{cases} \mathbf{a}_{11}\mathbf{x}_1 + \mathbf{a}_{12}\operatorname{imp}\mathbf{x}_2 + \ldots + \mathbf{a}_{1n}\operatorname{imp}\mathbf{x}_n \supseteq \mathbf{b}_1 \\ \mathbf{a}_{21}\operatorname{imp}\mathbf{x}_1 + \mathbf{a}_{22}\mathbf{x}_2 + \ldots + \mathbf{a}_{2n}\operatorname{imp}\mathbf{x}_n \supseteq \mathbf{b}_2 \\ \ldots \\ \mathbf{a}_{n1}\operatorname{imp}\mathbf{x}_1 + \mathbf{a}_{n2}\operatorname{imp}\mathbf{x}_2 + \ldots + \mathbf{a}_{nn}\mathbf{x}_n \supseteq \mathbf{b}_n \end{cases} \tag{5}$$

should be solved instead, where each occurrence of the variables except the diagonal ones is transformed to its proper or improper counterpart in the new system. The notations of Eq.(4) and Eq.(5) are simplified as

$$\mathbf{A}\cdot\mathbf{x}^{pro} \subseteq \mathbf{b} \tag{6}$$

and

$$\mathbf{A}\cdot\mathbf{x}^{imp} \supseteq \mathbf{b} \tag{7}$$

The algebraic solutions can be interpreted as

$$(\forall a_P \in \mathbf{a}_P)(\forall b_I \in \operatorname{pro}\mathbf{b}_I)(\forall x_P \in \mathbf{x}_P)(\exists a_I \in \operatorname{pro}\mathbf{a}_I)(\exists b_P \in \mathbf{b}_P)(\exists x_I \in \operatorname{pro}\mathbf{x}_I)(A\cdot x = b) \tag{8}$$

and

$$(\forall a_I \in \operatorname{pro}\mathbf{a}_I)(\forall b_P \in \mathbf{b}_P)(\forall x_I \in \operatorname{pro}\mathbf{x}_I)(\exists a_P \in \mathbf{a}_P)(\exists b_I \in \operatorname{pro}\mathbf{b}_I)(\exists x_P \in \mathbf{x}_P)(A\cdot x = b) \tag{9}$$

respectively.

An enhanced interpretable Jacobi algorithm is developed to solve Eq.(6), as listed in Figure 2, where the Jacobi operator is applied to the original and the transformed variables alternately. We define a *proper transform Jacobi interval operator* as

$$\mathfrak{I}^{pro}(\mathbf{x}_i) := \frac{\mathbf{b}_i - \sum_{i\neq j} \mathrm{dual}\,\mathbf{a}_{ij} \cdot \mathrm{imp}\,\mathbf{x}_j}{\mathrm{dual}\,\mathbf{a}_{ii}} \qquad \left(0 \notin \mathrm{pro}\,\mathbf{a}_{ii} \text{ and } i = 1,\cdots,n\right) \tag{10}$$

Applying the Jacobi operator in Eq.(3) to the transformed variable $\mathbf{x}^{pro}$ is equivalent to applying the proper transform Jacobi operator in Eq.(10) to the original variable $\mathbf{x}$.

Similarly, an interpretable Jacobi algorithm to solve Eq.(7) is listed in Figure 3, where an *improper transform Jacobi interval operator* is defined as

$$\mathfrak{I}^{imp}(\mathbf{x}_i) := \frac{\mathbf{b}_i - \sum_{i\neq j} \mathrm{dual}\,\mathbf{a}_{ij} \cdot \mathrm{pro}\,\mathbf{x}_j}{\mathrm{dual}\,\mathbf{a}_{ii}} \qquad \left(0 \notin \mathrm{pro}\,\mathbf{a}_{ii} \text{ and } i = 1,\cdots,n\right) \tag{11}$$

Theorem 3.2 (1) If $\mathbf{x}$ is a solution to $\mathbf{A}\cdot\mathbf{x} \supseteq \mathbf{b}$, then $\mathbf{x}$ is also a solution to $\mathbf{A}\cdot\mathbf{x}^{pro} \supseteq \mathbf{b}$. (2) If $\mathbf{x}$ is a solution to $\mathbf{A}\cdot\mathbf{x} \subseteq \mathbf{b}$, then $\mathbf{x}$ is also a solution to $\mathbf{A}\cdot\mathbf{x}^{imp} \subseteq \mathbf{b}$.

In Figure 2, at the ($2k$)th step in the iterative solving process, applying the Jacobi operator in Eq.(3) to $\mathbf{x}^{(2k-1)}$, we receive $\mathbf{x}^{(2k)} = \mathfrak{I}(\mathbf{x}^{(2k-1)})$ such that $\mathbf{A}\cdot\mathbf{x}^{pro(2k)} \supseteq \mathbf{A}\cdot\mathbf{x}^{(2k)} \supseteq \mathbf{b}$. Then at the ($2k+1$)th step, applying the proper transform Jacobi operator in Eq.(10) to $\mathbf{x}^{(2k)}$, we have $\mathbf{x}^{(2k+1)} = \mathfrak{I}(\mathbf{x}^{pro(2k)})$ such that $\mathbf{A}\cdot\mathbf{x}^{(2k+1)} \subseteq \mathbf{b}$. The iteration continues until stopping criteria are met. If $\mathbf{x}^{(k)}$ converges to $\mathbf{x}^{\infty}$, $\mathbf{x}^{\infty}$ is a solution to the interpretable linear system in Eq.(6). The iteration of the algorithm in Figure 3 is similar.

Input: generalized interval matrix $\mathbf{A} \in \mathbb{KR}^n \times \mathbb{KR}^n$,
generalized interval vector $\mathbf{b} \in \mathbb{KR}^n$
Output: generalized interval vector $\mathbf{x} \in \mathbb{KR}^n$

1. Initial estimation of $\mathbf{y}^{(0)}$ as the point (real) solution of $\mathrm{mid}(\mathbf{A})x = \mathrm{mid}(\mathbf{b})$ such that $\mathrm{imp}\,\mathbf{A}\cdot\mathbf{y}^{(0)} \subseteq \mathrm{pro}\,\mathbf{b}$;
2. $\mathbf{x}^{(0)} = \mathfrak{I}(\mathbf{y}^{(0)})$ associated with $\mathrm{imp}\,\mathbf{A}\cdot\mathbf{x} = \mathrm{pro}\,\mathbf{b}$, which is also the initial solution to $\mathbf{A}\cdot\mathbf{x} \supseteq \mathbf{b}$ as $k = 0$;
3. Iterate the following steps associated with $\mathbf{A}\cdot\mathbf{x} = \mathbf{b}$ until $\mathbf{x}$ converges:
   $k = k+1$;
   If k is odd, $\mathbf{x}^{(k)} = \mathfrak{I}^{pro}(\mathbf{x}^{(k-1)})$;
   otherwise, $\mathbf{x}^{(k)} = \mathfrak{I}(\mathbf{x}^{(k-1)})$.

Figure 2. Proper transform Jacobi algorithm for the linear system in Eq.(4)

Input: generalized interval matrix $\mathbf{A} \in \mathbb{KR}^n \times \mathbb{KR}^n$,
generalized interval vector $\mathbf{b} \in \mathbb{KR}^n$
Output: generalized interval vector $\mathbf{x} \in \mathbb{KR}^n$

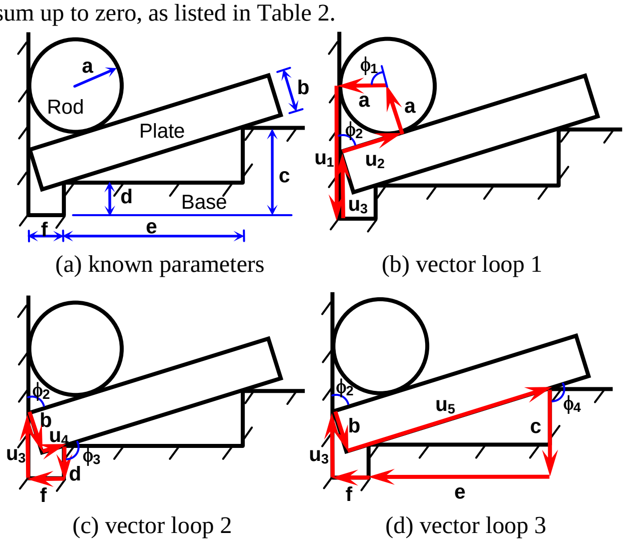
1. Initial estimation of $\mathbf{y}^{(0)}$ as the point (real) solution of $\mathrm{mid}(\mathbf{A})x = \mathrm{mid}(\mathbf{b})$ such that $\mathrm{pro}\,\mathbf{A}\cdot\mathbf{y}^{(0)} \supseteq \mathrm{imp}\,\mathbf{b}$;
2. $\mathbf{x}^{(0)} = \mathfrak{I}(\mathbf{y}^{(0)})$ associated with $\mathrm{pro}\,\mathbf{A}\cdot\mathbf{x} = \mathrm{imp}\,\mathbf{b}$, which is also the initial solution to $\mathbf{A}\cdot\mathbf{x} \subseteq \mathbf{b}$ as $k = 0$;
3. Iterate the following steps associated with $\mathbf{A}\cdot\mathbf{x} = \mathbf{b}$ until $\mathbf{x}$ converges:
   $k = k+1$;
   If k is odd, $\mathbf{x}^{(k)} = \mathfrak{I}^{imp}(\mathbf{x}^{(k-1)})$;
   otherwise, $\mathbf{x}^{(k)} = \mathfrak{I}(\mathbf{x}^{(k-1)})$.

Figure 3. Improper transform Jacobi algorithm for the linear system in Eq.(5)

The interpretable linear system solving algorithms in Figure 2 and Figure 3 ensure the interpretability of numerical results. This is regarded as an important step towards interpretable tolerance analysis. Its advantage of completeness and soundness assessment is illustrated in the example in Section 3.2.

3.2 Example A: stacked block assembly - linear

Figure 4 shows an example of a stacked block assembly including a base, a rectangular plate and a cylindrical rod. With the known size tolerances of manufactured components, the kinematic variations of the assembly can be calculated with three interval vector loops. Each of the closed loops defines the algebraic relations between the size and kinematic variations. The vector components in each 2D translational or rotational direction sum up to zero, as listed in Table 2.



(a) known parameters (b) vector loop 1
(c) vector loop 2 (d) vector loop 3

Figure 4. Stacked block assembly with closed loops of variations

Suppose that the limits of angle variations in the assembly are known, the tolerance analysis problem is reduced to solving a linear system. It can be solved using the improper transform Jacobi algorithm in Figure 3. As listed in Table 2, the size tolerances (a, b, c, d, e, f) and angular variations ($\phi_1, \phi_2, \phi_3, \phi_4$) are assigned to be improper while the estimated kinematic variations (u_1, u_2, u_3, u_4, u_5) are proper. Based on the interpretability rule in Eq.(9), the result is interpreted as: given the size tolerances (a, b, c, d, e, f) and the functionally critical kinematic variations ($\phi_1, \phi_2, \phi_3, \phi_4$), the non-functional kinematic variations (u_1, u_2, u_3, u_4, u_5) can be estimated. The most important inference from the interpretation is that the numerical estimations (u_1, u_2, u_3, u_4, u_5) are complete. Producing verifiable results is the main advantage of solving interpretable systems compared to the traditional analysis methods, where results are not interpretable and completeness or soundness of the estimations is unknown.

3.3 MIA direct linearization for nonlinear systems

When constraints of variations are nonlinear, a linearization process may be used to reduce the complexity of the direct computation of nonlinear systems. The linear approximation usually changes the semantic relationships among variables. Therefore, the numerical result is only interpretable with respect to the linearized system instead of the original nonlinear one.

To solve a nonlinear system

$$\mathbf{F}(\mathbf{s}, \mathbf{k}) = 0 \tag{12}$$

where $\mathbf{s} \in \mathbb{R}^m$ is a size variation vector, $\mathbf{k} \in \mathbb{R}^n$ is a kinematic variation vector, and $\mathbf{F} : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function, we can apply the Taylor's expansion to the nonlinear system with respect to the nominal values of $\mathbf{s}$ and $\mathbf{k}$. Then we have a linearized interval relation

$$\left[\frac{\partial F_l}{\partial s_i}\right]_{n\times m} \Delta\mathbf{s} + \left[\frac{\partial F_l}{\partial k_j}\right]_{n\times n} \Delta\mathbf{k} = 0 \quad (l = 1,\ldots,n; i = 1,\ldots,m; j = 1,\ldots,n) \tag{13}$$

where $\left[\frac{\partial F_l}{\partial s_i}\right] \in \mathbb{R}^{n\times m}$ and $\left[\frac{\partial F_l}{\partial k_j}\right] \in \mathbb{R}^{n\times n}$ are sensitivity matrices. Because the nominal values are used, the elements in the sensitivity matrices have real values. $\Delta\mathbf{s} \in \mathbb{KR}^m$ and $\Delta\mathbf{k} \in \mathbb{KR}^n$ are interval vectors corresponding to the size and kinematic variations respectively.

With the real sensitivity matrices in Eq.(13), tolerance accumulations can be estimated directly. For example, if the kinematic variations are to be calculated, we solve

$$\left[\frac{\partial F_l}{\partial k_j}\right]_{n\times n} \Delta\mathbf{k} = -\left[\frac{\partial F_l}{\partial s_i}\right]_{n\times m} \mathrm{dual}\,\Delta\mathbf{s} \tag{14}$$

The kinematic variations can be simply derived from

$$\Delta\mathbf{k} = -\left[\frac{\partial F_l}{\partial k_j}\right]_{n\times n}^{-1} \left[\frac{\partial F_l}{\partial s_i}\right]_{n\times m} \mathrm{dual}\,\Delta\mathbf{s} \tag{15}$$

Because of the linear approximation, the full semantics of the original nonlinear relations in Eq.(12) cannot be obtained directly from the numerical result. Instead, it is only interpretable with respect to Eq.(13). In parallel, the result is an algebraic solution of Eq.(13) instead of Eq.(12).

3.4 Example B: stacked block assembly - nonlinear

Suppose that the angle variations in the previous stacked block assembly example in Section 3.2 are unknown, the tolerance analysis problem is nonlinear. The variables are listed in Table 3. In this example, we assume the dimensional tolerances are a priori. That is, $\Delta\mathbf{s} \in \mathbb{IR}^6$ are proper as in Eq.(13). The numerical estimations of the kinematic variations $\Delta\mathbf{k} \in \overline{\mathbb{IR}}^9$ are shown in Table 4. Compared to the results from the DLM methods [37], which are purely numerical estimations, the kinematic variation intervals from the MIA direct linearization method are improper, in contrast to the size variations as proper intervals. The modality difference indicates the semantic difference between dimensional tolerances and kinematic variations, which are regulating, buffering, and more flexible. Furthermore, both completeness and soundness of the estimation from the linearized system can be verified from the interpretations based on the interpretability principles [5]. Therefore, the MIA direct linearization method provides more information than the regular numerical methods.

In general, different modality assignments of tolerance intervals lead to different numerical results and interpretations. Depending on the designer's intention and desired semantics, corresponding combinations of proper and improper intervals can be applied. For instance, if the dimensional tolerances a and b become non-critical or controllable, they are assigned to be proper. Then the new numerical estimations are: $u_1 = 18.7181 \pm 0.1374$, $u_2 = 8.6705 \pm 0.1641$, $u_3 = 10.0477 \mp 0.1311$, $u_4 = 2.1894 \mp 0.1629$, $u_5 = 27.2965 \mp 0.4727$, $\phi_1 = 74.7243 \mp 0.0108$, $\phi_2 = -74.7243 \mp 0.0108$, $\phi_3 = -105.2761 \mp 0.0108$, $\phi_4 = -105.2761 \mp 0.0108$, where u_1 and u_2 are a priori tolerances. The corresponding interpretation is

$$\begin{aligned}
&(\forall u_1 \in [18.5807, 18.8555])(\forall u_2 \in [8.5064, 8.8346])(\forall c \in [10.55, 10.8])\\
&(\forall d \in [3.91, 4.21])(\forall e \in [23.87, 24.57])(\forall f \in [3.78, 4.03])\\
&(\exists a \in [6.42, 6.82])(\exists b \in [6.73, 6.88])(\exists u_3 \in [9.9166, 10.1788])\\
&(\exists u_4 \in [2.0265, 2.3523])(\exists u_5 \in [26.8238, 27.7692])\\
&(\exists \phi_1 \in [74.7135, 74.7351])(\exists \phi_2 \in [-74.7351, -74.7135])\\
&(\exists \phi_3 \in [-105.2869, -105.2653])(\exists \phi_4 \in [-105.2869, -105.2653])\\
&\left[\frac{\partial F_l}{\partial s_i}\right]_{9\times 6} \begin{bmatrix} a & b & c & d & e & f \end{bmatrix}^T\\
&+ \left[\frac{\partial F_l}{\partial k_j}\right]_{9\times 9} \begin{bmatrix} u_1 & u_2 & u_3 & u_4 & u_5 & \phi_1 & \phi_2 & \phi_3 & \phi_4 \end{bmatrix}^T = 0
\end{aligned}$$

Table 2. Linear problem in stacked block assembly

Known Size variations	$a = 6.62 \mp 0.2 \quad b = 6.805 \mp 0.075 \quad c = 10.675 \mp 0.125$ $d = 4.06 \mp 0.15 \quad e = 24.22 \mp 0.35 \quad f = 3.905 \mp 0.125$
Known Kinematic variations	$\phi_1 = 74.7243 \mp 0.4281 \quad \phi_2 = -74.7243 \mp 0.4281 \quad \phi_3 = -105.2761 \mp 0.4281 \quad \phi_4 = -105.2761 \mp 0.4281$
Unknown Kinematic variations	$u_1 = 18.7181 \pm ? \quad u_2 = 8.6705 \pm ? \quad u_3 = 10.0477 \pm ? \quad u_4 = 2.1894 \pm ? \quad u_5 = 27.2965 \pm ?$
Loop 1	$\begin{cases} F_1 = u_2 \cos(90+\phi_2) + a\cos(180+\phi_2) + a\cos(180+\phi_1+\phi_2) = 0 \\ F_2 = u_3 + u_2 \sin(90+\phi_2) + a\sin(180+\phi_2) + a\sin(180+\phi_1+\phi_2) - u_1 = 0 \\ F_3 = 90+\phi_2+90+\phi_1+90+90-360 = 0 \end{cases}$
Loop 2	$\begin{cases} F_4 = b\cos(\phi_2) + u_4 \cos(\phi_2+90) + d\cos(\phi_2+90+\phi_3) - f = 0 \\ F_5 = u_3 + b\sin(\phi_2) + u_4 \sin(\phi_2+90) + d\sin(\phi_2+90+\phi_3) = 0 \\ F_6 = 90+\phi_2-90+90+\phi_3-90+180 = 0 \end{cases}$
Loop 3	$\begin{cases} F_7 = b\cos(\phi_2) + u_5 \cos(\phi_2+90) + c\cos(\phi_2+90+\phi_4) - e - f = 0 \\ F_8 = u_3 + b\sin(\phi_2) + u_5 \sin(\phi_2+90) + c\sin(\phi_2+90+\phi_4) = 0 \\ F_9 = 90+\phi_2-90+90+\phi_4-90+180 = 0 \end{cases}$
Linear equations	$\begin{cases} -u_1 + u_2 \sin(90+\phi_2) + u_3 + a\sin(180+\phi_2) = 0 \\ u_2 \cos(90+\phi_2) + a(\cos(180+\phi_2) - 1) = 0 \\ u_3 + u_4 \sin(\phi_2+90) + b\sin(\phi_2) - d = 0 \\ u_4 \cos(\phi_2+90) + b\cos(\phi_2) - f = 0 \\ u_5 \cos(\phi_2+90) + b\cos(\phi_2) - e - f = 0 \end{cases}$ $\begin{bmatrix} -1 & [0.2707,0.2562] & 1 & 0 & 0 \\ 0 & [0.9667,0.9626] & 0 & 0 & 0 \\ 0 & 0 & 1 & [0.2707,0.2562] & 0 \\ 0 & 0 & 0 & [0.9667,0.9626] & 0 \\ 0 & 0 & 0 & 0 & [0.9667,0.9626] \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \\ \mathbf{u}_4 \\ \mathbf{u}_5 \end{bmatrix} = \begin{bmatrix} [-6.5922,-6.1804] \\ [8.0652,8.6659] \\ [10.3888,10.8602] \\ [1.9179,2.3054] \\ [25.7879,26.8754] \end{bmatrix}$
Result of interpretable Jacobi algorithm	$\begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \\ \mathbf{u}_4 \\ \mathbf{u}_5 \end{bmatrix} = \begin{bmatrix} [18.0583,19.3812] \\ [8.3430,9.0026] \\ [9.7404,10.3520] \\ [1.9839,2.3950] \\ [26.6762,27.9196] \end{bmatrix}$
Interpretation of result	$(\forall a \in [6.42,6.82])(\forall b \in [6.73,6.88])(\forall c \in [10.55,10.8])(\forall d \in [3.91,4.21])(\forall e \in [23.87,24.57])$ $(\forall f \in [3.78,4.03])(\forall \phi_1 \in [74.2962,75.1524])(\forall \phi_2 \in [-75.1524,-74.2962])$ $(\forall \phi_3 \in [-105.7042,-104.8480])(\forall \phi_4 \in [-105.7042,-104.8480])(\exists u_1 \in [18.0583,19.3812])$ $(\exists u_2 \in [8.3430,9.0026])(\exists u_3 \in [9.7404,10.3520])(\exists u_4 \in [1.9839,2.3950])(\exists u_5 \in [26.6762,27.9196])$ $\begin{cases} -u_1 + u_2 \sin(90+\phi_2) + u_3 + a\sin(180+\phi_2) = 0 \\ u_2 \cos(90+\phi_2) + a(\cos(180+\phi_2) - 1) = 0 \\ u_3 + u_4 \sin(\phi_2+90) + b\sin(\phi_2) - d = 0 \\ u_4 \cos(\phi_2+90) + b\cos(\phi_2) - f = 0 \\ u_5 \cos(\phi_2+90) + b\cos(\phi_2) - e - f = 0 \end{cases}$

Table 3. The variation formulation of loops

Known size variations	$a = 6.62 \pm 0.2 \quad b = 6.805 \pm 0.075 \quad c = 10.675 \pm 0.125$ $d = 4.06 \pm 0.15 \quad e = 24.22 \pm 0.35 \quad f = 3.905 \pm 0.125$
Unknown kinematic variations	$u_1 = 18.7181 \pm ? \quad u_2 = 8.6705 \pm ? \quad u_3 = 10.0477 \pm ? \quad u_4 = 2.1894 \pm ? \quad u_5 = 27.2965 \pm ?$ $\phi_1 = 74.7243 \pm ? \quad \phi_2 = -74.7243 \pm ? \quad \phi_3 = -105.2761 \pm ? \quad \phi_4 = -105.2761 \pm ?$
Linearized system	$\left[\frac{\partial F_l}{\partial s_i}\right]_{9\times 6} = \begin{bmatrix} -1.2635 & 0 & 0 & 0 & 0 & 0 \\ 0.9647 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2635 & 0 & 0 & 0 & -1 \\ 0 & -0.9647 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2635 & 0 & 0 & -1 & -1 \\ 0 & -0.9647 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \ \Delta \mathrm{s} = \begin{bmatrix} [-0.2, 0.2] \\ [-0.075, 0.075] \\ [-0.125, 0.125] \\ [-0.15, 0.15] \\ [-0.35, 0.35] \\ [-0.125, 0.125] \end{bmatrix}$ $\left[\frac{\partial F_l}{\partial k_j}\right]_{9\times 9} = \begin{bmatrix} 0 & 0.9647 & 0 & 0 & 0 & 18.7181 & 10.0477 & 0 & 0 \\ -1 & 0.2635 & 1 & 0 & 0 & -6.62 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0.9647 & 0 & 0 & 10.0477 & 4.0600 & 0 \\ 0 & 0 & 1 & 0.2635 & 0 & 0 & 0 & -3.9050 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0.9647 & 0 & 10.0477 & 0 & 10.6750 \\ 0 & 0 & 1 & 0 & 0.2635 & 0 & 0 & 0 & -28.125 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$
Equations with row rearrangement	$\begin{bmatrix} -1 & 0.2635 & 1 & 0 & 0 & -6.62 & 0 & 0 & 0 \\ 0 & 0.9647 & 0 & 0 & 0 & 18.7181 & 10.0477 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0.2635 & 0 & 0 & 0 & -28.125 \\ 0 & 0 & 0 & 0.9647 & 0 & 0 & 10.0477 & 4.0600 & 0 \\ 0 & 0 & 0 & 0 & 0.9647 & 0 & 10.0477 & 0 & 10.6750 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0.2635 & 0 & 0 & 0 & -3.9050 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta \mathbf{u}_1 \\ \Delta \mathbf{u}_2 \\ \Delta \mathbf{u}_3 \\ \Delta \mathbf{u}_4 \\ \Delta \mathbf{u}_5 \\ \Delta \boldsymbol{\varphi}_1 \\ \Delta \boldsymbol{\varphi}_2 \\ \Delta \boldsymbol{\varphi}_3 \\ \Delta \boldsymbol{\varphi}_4 \end{bmatrix} = \begin{bmatrix} [0.1929, -0.1929] \\ [0.2527, -0.2527] \\ [0.1973, -0.1973] \\ [0.1447, -0.1447] \\ [0.4947, -0.4947] \\ [0.0000, -0.0000] \\ [0.0000, -0.0000] \\ [0.2223, -0.2223] \\ [0.0000, -0.0000] \end{bmatrix}$
First interpretation of the result	$(\forall a \in [6.42, 6.82])(\forall b \in [6.73, 6.88])(\forall c \in [10.55, 10.8])(\forall d \in [3.91, 4.21])(\forall e \in [23.87, 24.57])$ $(\forall f \in [3.78, 4.03])(\exists u_1 \in [18.1761, 19.2601])(\exists u_2 \in [8.2123, 9.1467])(\exists u_3 \in [9.734, 10.3614])$ $(\exists u_4 \in [1.9165, 2.4623])(\exists u_5 \in [26.7756, 27.8174])(\exists \phi_1 \in [74.7015, 74.7471])$ $(\exists \phi_2 \in [-74.7015, -74.7471])(\exists \phi_3 \in [-105.2989, -105.2533])(\exists \phi_4 \in [-105.2989, -105.2533])$ $\left[\frac{\partial F_l}{\partial s_i}\right]_{9\times 6} \begin{bmatrix} a & b & c & d & e & f \end{bmatrix}^T + \left[\frac{\partial F_l}{\partial k_j}\right]_{9\times 9} \begin{bmatrix} u_1 & u_2 & u_3 & u_4 & u_5 & \phi_1 & \phi_2 & \phi_3 & \phi_4 \end{bmatrix}^T = 0$
Second interpretation of the result	$(\forall u_1 \in [18.1761, 19.2601])(\forall u_2 \in [8.2123, 9.1467])(\forall u_3 \in [9.734, 10.3614])(\forall u_4 \in [1.9165, 2.4623])$ $(\forall u_5 \in [26.7756, 27.8174])(\forall \phi_1 \in [74.7015, 74.7471])(\forall \phi_2 \in [-74.7015, -74.7471])$ $(\forall \phi_3 \in [-105.2989, -105.2533])(\forall \phi_4 \in [-105.2989, -105.2533])(\exists a \in [6.42, 6.82])$ $(\exists b \in [6.73, 6.88])(\exists c \in [10.55, 10.8])(\exists d \in [3.91, 4.21])(\exists e \in [23.87, 24.57])(\exists f \in [3.78, 4.03])$ $\left[\frac{\partial F_l}{\partial s_i}\right]_{9\times 6} \begin{bmatrix} a & b & c & d & e & f \end{bmatrix}^T + \left[\frac{\partial F_l}{\partial k_j}\right]_{9\times 9} \begin{bmatrix} u_1 & u_2 & u_3 & u_4 & u_5 & \phi_1 & \phi_2 & \phi_3 & \phi_4 \end{bmatrix}^T = 0$

Table 4. Comparison of MIA linearization and DLM

MIA Linearization	DLM Worst-Case	DLM Statistical
$\Delta\mathbf{u}_1 = [0.5420,\ -0.5420]$	$\Delta\mathbf{u}_1 = [-0.5421,\ 0.5421]$	$\Delta\mathbf{u}_1 = [-0.2998,\ 0.2998]$
$\Delta\mathbf{u}_2 = [0.4672,\ -0.4672]$	$\Delta\mathbf{u}_2 = [-0.3899,\ 0.3899]$	$\Delta\mathbf{u}_2 = [-0.2725,\ 0.2725]$
$\Delta\mathbf{u}_3 = [0.3137,\ -0.3137]$	$\Delta\mathbf{u}_3 = [-0.2942,\ 0.2942]$	$\Delta\mathbf{u}_3 = [-0.1844,\ 0.1844]$
$\Delta\mathbf{u}_4 = [0.2729,\ -0.2729]$	$\Delta\mathbf{u}_4 = [-0.2384,\ 0.2384]$	$\Delta\mathbf{u}_4 = [-0.1411,\ 0.1411]$
$\Delta\mathbf{u}_5 = [0.5209,\ -0.5209]$	$\Delta\mathbf{u}_5 = [-0.5174,\ 0.5174]$	$\Delta\mathbf{u}_5 = [-0.3836,\ 0.3836]$
$\Delta\boldsymbol{\varphi}_1 = [0.0228,\ -0.0228]$	$\Delta\boldsymbol{\varphi}_1 = [-0.8156,\ 0.8156]$	$\Delta\boldsymbol{\varphi}_1 = [-0.4784,\ 0.4784]$
$\Delta\boldsymbol{\varphi}_2 = [0.0228,\ -0.0228]$	$\Delta\boldsymbol{\varphi}_2 = [-0.8156,\ 0.8156]$	$\Delta\boldsymbol{\varphi}_2 = [-0.4784,\ 0.4784]$
$\Delta\boldsymbol{\varphi}_3 = [0.0228,\ -0.0228]$	$\Delta\boldsymbol{\varphi}_3 = [-0.8156,\ 0.8156]$	$\Delta\boldsymbol{\varphi}_3 = [-0.4784,\ 0.4784]$
$\Delta\boldsymbol{\varphi}_4 = [0.0228,\ -0.0228]$	$\Delta\boldsymbol{\varphi}_4 = [-0.8156,\ 0.8156]$	$\Delta\boldsymbol{\varphi}_4 = [-0.4784,\ 0.4784]$

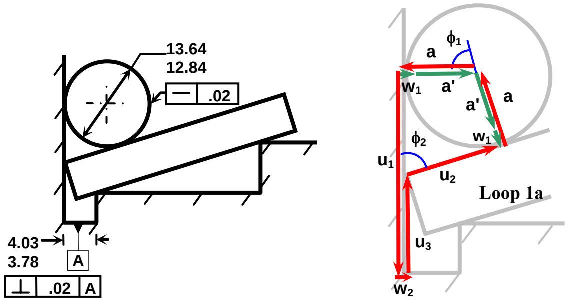
4. Geometric Tolerances

Size and geometric tolerances cannot always be stacked up independently in interval vector loops. Interdependency between size and geometric tolerances exists. For instance, the Envelope Rule does not allow elements of specified features to go beyond size limits. Size limits control the allowable magnitudes of geometric form variations. This implies that the variation of geometric form is decreased when the actual size approaches the Maximum Material Condition (MMC). Features would be required to have perfect form and the geometric variation is reduced to zero at the MMC. Therefore, the mating envelope or virtual condition of the feature is its size limit. However, if a geometric tolerance is applied to a feature of size, the Envelope Rule is overridden. Then the geometric tolerance can be stacked with the size tolerance independently to estimate the accumulative effect.

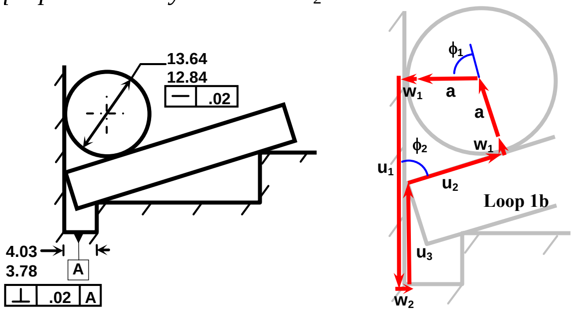
Depending on the relationship between size and geometric tolerances, interval vector loops need to be constructed in different ways. For example, in Figure 5-(a), the straightness tolerance of the rod is applied to the feature. The Envelope Rule is applied. The form variation of the rod $\mathbf{w}_1$ should not be included in the interval vector loop when estimating kinematic variations. The limit of the form variation $\mathbf{w}_1$ has been incorporated in the size tolerance $\mathbf{a}$. The size and geometric tolerances are not independent. In contrast, in Figure 5-(b), the straightness tolerance $\mathbf{w}_1$ is applied to a feature of size. The size and geometric tolerances thus are stacked up independently and must be included in the vector loop. Similarly, the perpendicularity tolerance $\mathbf{w}_2$ applied to the base is stacked up independently in the same figure. In Figure 5-(c), the flatness tolerance $\mathbf{w}_3$ is incorporated in the size tolerance $\mathbf{b}$. In Figure 5-(d), the perpendicularity tolerance $\mathbf{w}_4$ is incorporated in the size tolerance $\mathbf{e}$. The new formulation of vector loops with the consideration of geometric tolerances is shown in Table 5. Notice that the respective nominal values of form tolerances ($\mathbf{w}_1$ and $\mathbf{w}_3$) and orientation tolerances ($\mathbf{w}_2$ and $\mathbf{w}_4$) in the loops are different. Those of the orientation tolerances are zeros and the corresponding interval vectors are bidirectional. To estimate the variation accumulations in Table 5, the numerical methods developed in Section 3 can be applied.

Table 5. Formulation of closed loops with size, geometric, and kinematic variations

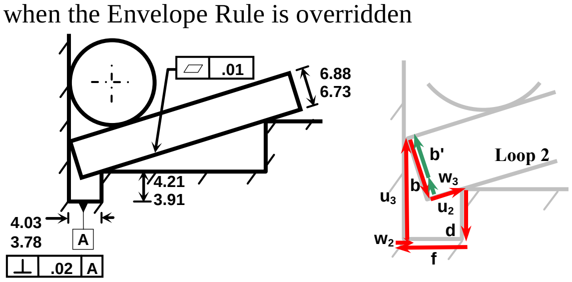
Known size variations	$a = 6.62 \pm 0.2 \quad b = 6.805 \pm 0.075 \quad c = 10.675 \pm 0.125$ $d = 4.06 \pm 0.15 \quad e = 24.22 \pm 0.35 \quad f = 3.905 \pm 0.125$
Known geometric variations	$w_1 = 0.01 \pm 0.01 \quad w_2 = 0 \pm 0.01 \quad w_3 = 0.01 \pm 0.01 \quad w_4 = 0 \pm 0.01$
Unknown kinematic variations	$u_1 = 18.7181 \pm ? \quad u_2 = 8.6705 \pm ? \quad u_3 = 10.0477 \pm ? \quad u_4 = 2.1894 \pm ? \quad u_5 = 27.2965 \pm ?$ $\phi_1 = 74.7243 \pm ? \quad \phi_2 = -74.7243 \pm ? \quad \phi_3 = -105.2761 \pm ? \quad \phi_4 = -105.2761 \pm ?$
Loop 1a (when the Envelope Rule is applied)	$\begin{cases} F_1 = w_2 + u_2\cos(90+\phi_2) + a\cos(180+\phi_2) + a\cos(180+\phi_1+\phi_2) = 0 \\ F_2 = u_3 + u_2\sin(90+\phi_2) + a\sin(180+\phi_2) + a\sin(180+\phi_1+\phi_2) - u_1 = 0 \\ F_3 = 90+\phi_2+90+\phi_1+90+90-360 = 0 \end{cases}$
Loop 1b (when the Envelope Rule is overridden)	$\begin{cases} F_1 = w_2 + u_2\cos(90+\phi_2) + (a+w_1)\cos(180+\phi_2) + (a+w_1)\cos(180+\phi_1+\phi_2) = 0 \\ F_2 = u_3 + u_2\sin(90+\phi_2) + a\sin(180+\phi_2) + a\sin(180+\phi_1+\phi_2) - u_1 = 0 \\ F_3 = 90+\phi_2+90+\phi_1+90+90-360 = 0 \end{cases}$
Loop 2	$\begin{cases} F_4 = w_2 + b\cos(\phi_2) + u_4\cos(\phi_2+90) + d\cos(\phi_2+90+\phi_3) - f = 0 \\ F_5 = u_3 + b\sin(\phi_2) + u_4\sin(\phi_2+90) + d\sin(\phi_2+90+\phi_3) = 0 \\ F_6 = 90+\phi_2-90+90+\phi_3-90+180 = 0 \end{cases}$
Loop 3	$\begin{cases} F_7 = w_2 + b\cos(\phi_2) + u_5\cos(\phi_2+90) + c\cos(\phi_2+90+\phi_4) - e - f = 0 \\ F_8 = u_3 + b\sin(\phi_2) + u_5\sin(\phi_2+90) + c\sin(\phi_2+90+\phi_4) = 0 \\ F_9 = 90+\phi_2-90+90+\phi_4-90+180 = 0 \end{cases}$



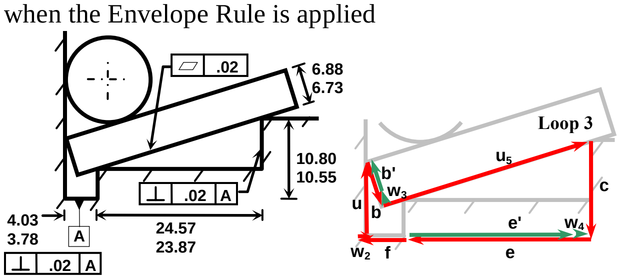
(a) The *straightness* tolerance $\mathbf{w}_1$ is not included in the loop when the Envelope Rule is applied, while the *perpendicularity* tolerance $\mathbf{w}_2$ is included



(b) The *straightness* tolerance $\mathbf{w}_1$ is included in the loop when the Envelope Rule is overridden



(c) The *flatness* tolerance $\mathbf{w}_3$ is not included in the loop when the Envelope Rule is applied



(d) The *perpendicularity* tolerance $\mathbf{w}_4$ is not included in the loop when the Envelope Rule is applied

Figure 5. Incorporating geometric tolerances in closed-loop analysis

5. Concluding Remarks

The purpose of semantic tolerance model is to enrich tolerance modeling and analysis structures such that more process-oriented tolerancing semantics and intents can be embedded in mathematical representations. The ultimate goal is to support better design and manufacturing specifications. In this paper, we presented a tolerance analysis approach based on interval vector loops. To ensure the interpretability of numerical results, interpretable Jacobi algorithms are developed to solve linear systems. Thus interpretable relations among variables can be maintained during computation. Nonlinear systems can be linearlized and variations can be estimated. With the interpretable relations, completeness and soundness of numerical results from linear systems can be verified. Producing verifiable results is the main advantage of solving interpretable systems compared to the traditional analysis methods, where results are not interpretable and completeness or soundness of the estimations is unknown.

Based on the algebraic closure property, we formulate form closure constraints of small displacement with closed loops of interval vectors. Geometric tolerances can also be included in the loops. Depending on the interdependency between size and geometric tolerances, form variations may be stacked up differently. The new approach enhances numerical analysis methods by ensuring algebraic closure and interpretability.

Future work may include developing interpretable nonlinear system solving methods. Since linearization is necessary to solve nonlinear systems numerically, maintaining the original logic relationships among variables during the process is critical. These new interpretable methods are expected to provide more accurate tolerance analysis. Since the developed numerical methods are generic in nature, they could potentially be applied in other engineering domains such as robust control and prediction under uncertainties.

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